Calculus and mathematical analysis

MATH1050

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- 10 credits
- Taught Semester 1
- Year running 2008/09
- Pre-requisites: A-level Mathematics
- Objectives: Calculate the derivatives and integrals of elementary functions. Do arithmetic calculations with complex numbers, including calculation of n-th roots. Calculate limits of simple sequences. Use different tests to study the convergence of infinite series. Compute Taylor power series.
- Syllabus: Differentiation; Hyperbolic functions and their inverses: properties and derivatives; Integration; Complex numbers: definition, de Moivre's theorem, the logarithmic function; Sequences: definition and calculation of limits; Infinite series; Taylor series.
- Form of teaching
  Lectures: 22 hours
- Form of assessment
  5 Examples (0%). 4 Workshop quizzes (15%). One 2 hour examination at end of semester (85%). Your answers to the examples will be returned to you with comments but they do not count towards the final grade of the module.
- Dates for handing in Examples
  Examples 1: 8 October. Examples 2: 22 October. Examples 3: 5 November. Examples 4: 19 November. Examples 5: 3 December.
- Dates for Workshop quizzes
  Quiz 1: 20 October. Quiz 2: 3 November. Quiz 3: 17 November. Quiz 4: 1 December.
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Chapter 1

Introduction

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1.1 Motivation

Why do we study calculus?
Calculus is an important branch of mathematics and is concerned with two basic operations called **differentiation** and **integration**. These operations are related and both rely for their definitions on the use of **limits**.
We study calculus with some revision of A-level work. We introduce the basic concepts of mathematical analysis.

1.2 Basics

Notation: We use the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ for the set of natural numbers, integers and real numbers, respectively:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\} \qquad \mathbb{Z} = \{0, 1, 2, 3, \dots\} \cup \{-1, -2, -3, -4 \dots\}$$

Given two real numbers $a < b$ we use the following interval notations:

$$[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\},$$

$$[a, b) = \{x \in \mathbb{R} | a \leq x < b\},$$

$$(a, b] = \{x \in \mathbb{R} | a < x \leq b\},$$

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}.$$

We also use the notations

$$[a, +\infty) = \{x \in \mathbb{R} | a \leq x\},$$

$$(a, +\infty) = \{x \in \mathbb{R} | a < x\},$$

$$(-\infty, a] = \{x \in \mathbb{R} | x \leq a\},$$

$$(-\infty, a) = \{x \in \mathbb{R} | x < a\}.$$

Functions: We shall use notations like

$$f : [a, b] \longrightarrow \mathbb{R}$$

to convey that f is a real-valued function defined for all $a \leq x \leq b$, *i.e.,* $f(x)$ is a real number for all $x \in [a, b]$. In general, $g : A \longrightarrow B$ conveys that g is a function with **domain** A and **range** B, *i.e.,* for all $x \in A$, $g(x) \in B$. $g(x)$ is the **value of** g **at** x.

Definition (provisional): A function is a rule which assigns, to each real number in its domain, some real number in its range.

1.2.1 Examples of functions

1. $f_1 : \mathbb{R} \longrightarrow \mathbb{R}, f_1(x) = x$

2. $f_2 : [0, +\infty) \longrightarrow [0, +\infty), f_2(x) = \sqrt{x}$

3. $f_3 : (0, +\infty) \longrightarrow (0, +\infty), f_3(x) = \frac{1}{x}$

4. $$f_4 : \mathbb{R} \longrightarrow \mathbb{R}, f_4(x) = \begin{cases} 0 & if \quad x \quad \text{is rational} \\ 2 & if \quad x \quad \text{is irrational} \end{cases}$$

5. $$f_5 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}, f_5(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

Building new functions: Suppose

$$f, g : A \longrightarrow \mathbb{R}.$$

Define

$$f + g : A \longrightarrow \mathbb{R}$$

by $(f+g)(x) = f(x) + g(x)$, the **sum** of f and g.

$$f - g : A \longrightarrow \mathbb{R}$$

by $(f-g)(x) = f(x) - g(x)$, the **difference** of f and g.

$$(f \cdot g) : A \longrightarrow \mathbb{R}$$

by $(f \cdot g)(x) = f(x) \cdot g(x)$, the **product** of f and g.
If $g(x) \neq 0$ for all $x \in A$, we also define

$$\left(\frac{f}{g}\right) : A \longrightarrow \mathbb{R}$$

by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, the **quotient** of f and g.

1.2.2 Examples

$$f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^2$$

$$g : \mathbb{R} \longrightarrow \mathbb{R}, g(x) = \frac{1}{1+x^2}$$

$(f \cdot g)(x) = f(x) \cdot g(x) = x^2 \cdot \frac{1}{1+x^2} = \frac{x^2}{1+x^2}$, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2}{\frac{1}{1+x^2}} = x^2(1+x^2) = x^4 + x^2.$

1.3 Preliminary algebra

1.3.1 Polynomial functions

Definition: A polynomial of degree n is a function $f(x)$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

Here n is an integer $n > 0$, called the degree of the polynomial f.

Definition: A polynomial equation is an equation of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0.$$

It is satisfied by particular values of x, called the roots of the polynomial.
For $n = 1$, (linear case) we have

$$a_1 x + a_0 = 0 \Rightarrow x = -\frac{a_0}{a_1}.$$

For $n = 2$, (quadratic case) we have

$$a_2 x^2 + a_1 x + a_0 = 0 \Rightarrow x_{\pm} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}.$$

Theorem 1.1: An n-th degree polynomial equation has exactly n roots.

1.3.2 Factorising polynomials

We have just seen that a polynomial equation can be written in any of the following alternative forms

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \\ f(x) &= a_n (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_r)^{m_r}, \\ f(x) &= a_n (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n), \end{aligned}$$

with $m_1 + m_2 + \cdots + m_r = n$.

Example 1.1: The roots of a quadratic polynomial $f(x) = a_2x^2 + a_1x + a_0 = 0$ are α_1 and α_2, such that

$$\alpha_1 + \alpha_2 = -\frac{a_1}{a_2} \qquad \alpha_1 \cdot \alpha_2 = \frac{a_0}{a_2}.$$

1.3.3 Trigonometric identities

Single-angle identities:

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1, \\ 1 + \tan^2\theta &= \sec^2\theta, \\ \cot^2\theta + 1 &= \csc^2\theta. \end{aligned}$$

Compound-angle identities:

$$\begin{aligned} \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B, \\ \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B. \end{aligned}$$

Double-angle identities:

$$\begin{aligned} \cos(2A) &= 1 - 2\sin^2 A, \\ \sin(2A) &= 2\sin A \cos A, \end{aligned}$$

1.4 Coordinate geometry

Equation of a straight-line: The standard form for a straight-line graph is

$$y = mx + c,$$

representing a linear relationship between the independent variable x and the dependent variable y. The slope m is equal to the tangent of the angle the line makes with the x-axis and c is the intercept of the y-axis. An alternative form for the equation of a straight line is

$$ax + by + k = 0,$$

with $m = -\frac{a}{b}$ and $c = -\frac{k}{b}$. This form treats x and y on a more symmetrical basis, the intercepts on the two axes being $-\frac{k}{a}$ and $-\frac{k}{b}$, respectively.

Equation of a line that passes through (x_1, y_1) and (x_2, y_2): Given two points (x_1, y_1) and (x_2, y_2), we find the equation of the line that passes through both of them as follows. The slope is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and

$$y - y_1 = m(x - x_1)$$

or

$$y - y_2 = m(x - x_2)$$

is the desired linear relationship between y and x.

Example 1.2: Find the equation of the line that passes through the points $(1, 2)$ and $(5, 3)$.
`Do as an exercise at home.`

1.5 Partial fractions

Example 1.3: Express the function

$$f(x) = \frac{4x + 2}{x^2 + 3x + 2}$$

in partial fractions.
We write

$$f(x) = \frac{g(x)}{h(x)} = \frac{4x + 2}{x^2 + 3x + 2}.$$

In this case the denominator $h(x)$ has zeros at $x = -1$ and $x = -2$; Thus the partial fraction expansion will be of the form

$$f(x) = \frac{4x + 2}{x^2 + 3x + 2} = \frac{A_1}{x + 1} + \frac{A_2}{x + 2} \Rightarrow 4x + 2 = A_1(x + 2) + A_2(x + 1) \Rightarrow A_1 = -2 \qquad A_2 = 6.$$

Example 1.4: If the denominator has repeated factors, the expansion is carried out as follows:

$$f(x) = \frac{x - 4}{(x + 1)(x - 2)^2} = \frac{A}{x + 1} + \frac{Bx + C}{(x - 2)^2} \Rightarrow A = -\frac{5}{9} \qquad B = \frac{5}{9} \qquad C = -\frac{16}{9}.$$

1.6 Binomial expansion

We consider the general expansion of $f(x) = (x + y)^n$, where x and y may stand for constants, variables or functions and n is a positive integer.

$$\begin{aligned}(x+y)^1 &= x+y,\\ (x+y)^2 &= (x+y)(x+y) = x^2+2xy+y^2,\\ (x+y)^3 &= (x+y)(x^2+2xy+y^2) = x^3+3x^2y+3xy^2+y^3.\end{aligned}$$

The general expression, the binomial expansion for power n is given by

$$(x+y)^n = \sum_{m=0}^{n} C_{n,m} x^{n-m} y^m,$$

where

$$C_{n,m} \equiv \frac{n!}{m!(n-m)!} \equiv \binom{n}{m}.$$

1.7 The principle of mathematical induction (MI or mi)

Suppose $P(k)$ means that the property P holds for the integer k.
Suppose that m is an integer and that
(1) $P(m)$ is true,
(2) Whenever $k \geq m$ and $P(k)$ is true, $P(k+1)$ is true.
Then $P(n)$ is true for all integers $n \geq m$.
(1) is called the **Base** and (2) is called the **Induction Step**. The supposition that $P(k)$ is true is called the **inductive hypothesis**.

Example: Sum of the first n integers.

$$1+2+\cdots+n = \frac{n(n+1)}{2} \qquad\qquad P(n)$$

Proof by MI:
Base: $P(1)$ is true as $1 = \frac{1(1+1)}{2}$.
Induction step: Let $k \geq 1$ and suppose $P(k)$ is true.
Thus $1+2+\cdots+k = \frac{k(k+1)}{2}$.
Hence $1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+1+1)}{2}$, and therefore $P(k+1)$ is true as well.

Problem: Let $P(n)$ stand for $''6^{2n}-1''$ is divisible by 35.
Claim: $P(n)$ is true for all integers $n \geq 1$.
Proof by MI:
Base: $P(1)$ is true as $6^{2n}-1 = 6^{2\cdot 1}-1 = 36-1 = 35$.

Induction step: Let $k \geq 1$ and suppose $P(k)$ is true. Thus $6^{2k} - 1$ is divisible by 35. Hence

$$6^{2(k+1)} - 1 = 6^{2k+2} - 1 = 6^{2k} \cdot 36 - 1 = 6^{2k} \cdot (35 + 1) - 1 = (6^{2k} - 1) + 6^{2k} \cdot 35,$$

which is divisible by 35, and therefore $P(k + 1)$ is true as well.

Exercise: Let $P(n)$ stand for $''n^4 + 2n^3 + 2n^2 + n''$ is divisible by 6.
Claim: $P(n)$ is true for all integers $n \geq 1$.
Proof by MI: `Do as an exercise at home.`

Theorem 1.2: If $x \neq 1$, then

$$\sum_{m=0}^{n} x^m = \frac{1 - x^{n+1}}{1 - x} \quad \text{for all integers} \quad n \geq 0.$$

Proof by MI: `Do as an exercise at home.`

Chapter 2

Limits

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2.1 Provisional definition

2.1.1

The function f will be said to have the **limit** L as x tends to a, if when x is arbitrarily close to, but unequal to a, $f(x)$ is arbitrarily close to L.
The statement "tends to a" is written as $x \to a$, and when the limit of $f(x)$ exists as $x \to a$, this will be shown by writing

$$\lim_{x \to a} f(x) = L.$$

Example 1: Let $f(x) = x \sin \frac{1}{x}$.

$$f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$$

What is $\lim_{x \to 0} f(x)$?

$$\left| \sin \frac{1}{x} \right| \leq 1 \qquad \forall x \neq 0.$$

Take x close to 0 but $x \neq 0$. If $0 < x < 1/10$ this means

$$|f(x)| = \left| x \cdot \sin \frac{1}{x} \right| = |x| \cdot \left| \sin \frac{1}{x} \right| \Rightarrow |f(x)| \leq |x| < \frac{1}{10} \Rightarrow \lim_{x \to 0} f(x) = 0.$$

Example 2: Let $g(x) = x^2$.

$$g : \mathbb{R} \to \mathbb{R}$$

Claim: $\lim_{x\to a} g(x) = a^2$.
Proof:

$$|g(x) - a^2| = |x^2 - a^2| = |(x-a)(x+a)| = |x-a| \cdot |x+a|.$$

If $|x - a| < 1/n$ with $n \geq 1$ then

$$|x+a| \leq |x| + |a| \leq |a| + 1 + |a| = 2 \cdot |a| + 1 \Rightarrow |g(x) - a^2| \leq (2 \cdot |a| + 1) \cdot \frac{1}{n}.$$

We can choose n large enough, so that $g(x)$ gets arbritrarily close to a^2. This shows that $\lim_{x\to a} g(x) = a^2$.

Example 3: Let

$$h : \mathbb{R} \longrightarrow \mathbb{R}, f_4(x) = \begin{cases} 0 & if \quad x \quad \text{is rational} \\ 2 & if \quad x \quad \text{is irrational} \end{cases}$$

Then $\lim_{x\to a} h(x)$ does not exist.

2.1.2 Elementary properties of limits

Suppose

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M.$$

Then we have
(1) $\lim_{x\to a}[b \cdot f(x)] = b \cdot L$, with $b \in \mathbb{R}$.

(2) $\lim_{x\to a}[f(x) \pm g(x)] = L \pm M$.

(3) $\lim_{x\to a}[f(x) \cdot g(x)] = L \cdot M$.

(4) If $M \neq 0$ $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$.

Example 4: Find $\lim_{x\to 2} \left[\frac{x^2+5x+3}{2x^3-x+4}\right]$. In this case, set $x = 2$ and get

$$\lim_{x\to 2} \left[\frac{x^2+5x+3}{2x^3-x+4}\right] = \left[\frac{2^2+5\cdot 2+3}{2\cdot 2^3-2+4}\right] = \frac{17}{18}.$$

Example 5: Find $\lim_{x\to 1}\left[\frac{2x^2+x-3}{x^2+x-2}\right]$. If we set $x = 1$, we get

$$\left[\frac{2\cdot 1^2+1-3}{1^2+1-2}\right] = \frac{0}{0}.$$

We realise

$$\lim_{x\to 1}\left[\frac{2x^2+x-3}{x^2+x-2}\right] = \lim_{x\to 1}\left[\frac{(x-1)(2x+3)}{(x-1)(x+2)}\right] = \lim_{x\to 1}\left[\frac{(2x+3)}{(x+2)}\right] = \frac{5}{3}.$$

2.2 Continuity

If f is an arbitrary function, it is not necessarily true that

$$\lim_{x\to a} f(x) = f(a).$$

Definition: The function f is **continuous at** a if

$$\lim_{x\to a} f(x) = f(a).$$

2.2.1 Examples

1. The function $g(x) = x^2$ is continuous everywhere.

2. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 2 \\ 2 & \text{if} \quad x = 2 \\ 3 & \text{if} \quad x > 2 \end{cases}$$

   f is continuous at x for all $x < 2$ and $x > 2$, but <u>not</u> continous at $x = 2$.

3. Define $h : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if} \quad x \neq 2 \\ 4 & \text{if} \quad x = 2 \end{cases}$$

   Is h continuous at $x = 2$? We compute the following limit

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4 = h(2).$$

   We can say h is continuous at $x = 2$.

Chapter 3

Differentiation

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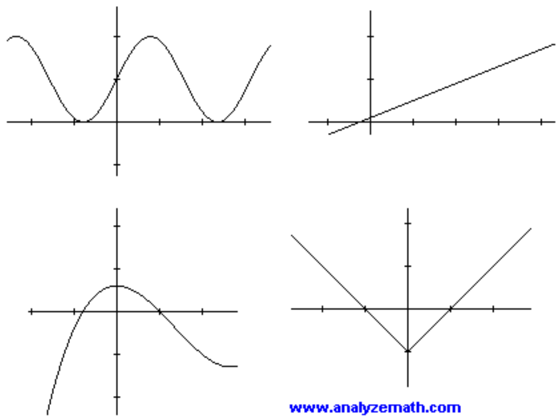
3.1 Intuitive definition

Here are the graphs of some continuous functions.



3.1.1 Examples of different behaviour

(1) $f(x) = |x|$
(2) $f(x) = \sqrt{|x|}$
(3)

$$f(x) = \begin{cases} x & \text{if} \quad x \geq 0 \\ x^2 & \text{if} \quad x < 0 \end{cases}$$

These functions show certain types of misbehaviour at $(0,0)$. They are "bent" at $(0,0)$, unlike the graph of the function in the following picture:

$$f(x) = x^3 - 4x^2 + 2x \ .$$

"Bent" at $(0,0)$ means a "tangent line" to the graph cannot be drawn. How can we define the notion of a tangent line to a point in the graph of a function? A tangent line cannot be defined as a line which intersects the graph only once

If $h \neq 0$, then the two distinct points $(a, f(a))$ and $(a+h, f(a+h))$ determine a straight line, whose slope is

$$\frac{f(a+h) - f(a)}{h}.$$

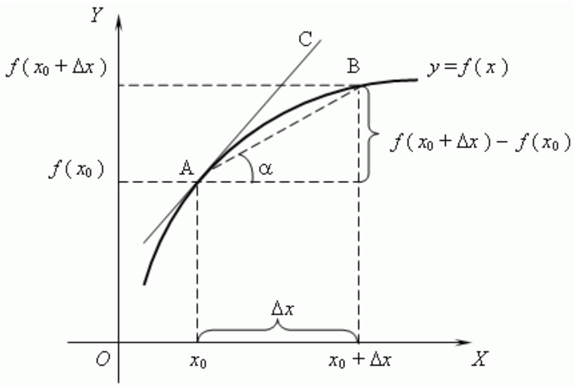


Fig. 1

We have never before talked about a "limit" of lines, but we can talk about the limit of their slopes: the slope of the tangent line through $(a, f(a))$ should be

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

We are ready now for a **definition** and some comments.

3.2 Differentiation

Let us consider a real-valued function $f : I \to \mathbb{R}$, with I a real interval, that is $I \subset \mathbb{R}$. Let us also consider a point in that interval, $a \in I$. We define

Definition: The function $f : I \to \mathbb{R}$ is differentiable at $a \in I$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. In this case the limit is denoted by $f'(a)$ and is called the derivative of f at a. We also say that f is differentiable if f is differentiable at a for every a in the domain of f.

We define the tangent line to the graph of f at $(a, f(a))$ to be the line through $(a, f(a))$ with slope $f'(a)$. This means that the tangent line at $(a, f(a))$ is defined only if f is differentiable at a.
We denote by f' the function whose domain is the set of all numbers a such that f is differentiable at a, and whose value at such a number a is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

3.2.1 Examples

1. The constant function $f(x) = c$. We have $f'(x) = 0$ for all x.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

2. The linear function $f(x) = c\cdot x + d$. We have $f'(x) = c$ for all x.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c(x+h)-c\cdot x}{h} = \lim_{h\to 0}\frac{cx+ch-cx}{h} = \lim_{h\to 0}\frac{ch}{h} = c.$$

3. The quadratic function $f(x) = x^2$. We have $f'(x) = 2x$ for all x.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{2hx+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

Theorem 3.1: If f is a constant function, $f(x) = c$, then $f'(x) = 0$ for all $x \in \mathbb{R}$.
Proof: We already showed that. □

Theorem 3.2: If f and g are differentiable at a, then $f + g$ is also differentiable at a, and

$$(f+g)'(a) = f'(a) + g'(a).$$

Proof:

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h} = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)+g'(x).\end{aligned}$$

Theorem 3.3: If f and g are differentiable at a, then $f \cdot g$ is also differentiable at a, and

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a).$$

Proof:

$$\begin{aligned}
(f \cdot g)'(x) &= \lim_{h \to 0} \frac{(f \cdot g)(x+h) - (f \cdot g)(x)}{h} = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x+h) + f(x) \cdot g(x+h) - f(x) \cdot g(x)}{h} \\
&= \lim_{h \to 0} \left[g(x+h) \cdot \frac{f(x+h) - f(x)}{h} \right] + \lim_{h \to 0} \left[f(x) \cdot \frac{g(x+h) - g(x)}{h} \right] \\
&= g(x) f'(x) + f(x) g'(x).
\end{aligned}$$

In the last step we have made use of the fact that both f and g are differentiable and therefore continuous. This implies

$$\lim_{h \to 0} g(x+h) = g(x) \; , \quad \lim_{h \to 0} f(x) = f(x) \; .$$

Lemma 3.4: If $g(x) = c \cdot f(x)$ and f is differentiable at a, then g is differentiable at a, and

$$g'(a) = c \cdot f'(a).$$

Proof: `Do at home as an exercise.`

To demonstrate what we have already achieved, we will compute the derivative of some more special functions.

Theorem 3.5: If $f(x) = x^n$ for some integer $n \geq 1$, then $f'(x) = nx^{n-1}$. for all x.
Proof: The proof will be by induction on n.

Base: if $n = 1$, $f(x) = x$ and we know that $f'(x) = 1$.
We check with the "proposed rule" (right-hand-side).

$$f'(x) = nx^{n-1} \quad \text{for} \quad n = 1 \Rightarrow f'(x) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \; .$$

We have verified that **YES** the proposed rule gives the correct answer

Induction Step: we assume that it is true for $n \geq 1$. That is, if $f(x) = x^n$, then its derivative is given by $f'(x) = nx^{n-1}$.
We need to show that the "proposed rule" is correct for $n + 1$. This is as follows:
If $f(x) = x^{n+1}$ we can make use of the product rule as follows

$$f(x) = x^{n+1} = x^n \cdot x = g(x) \cdot h(x) \Rightarrow f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x) \; ,$$

with

$$g(x) = x^n \qquad \text{and} \qquad h(x) = x \; .$$

Notice that from the base step we know the derivative of $h(x)$ and from the induction step we know the derivative of $g(x)$. We conclude then

$$\Rightarrow f'(x) = nx^{n-1} \cdot x + x^n \cdot 1 = nx^n + x^n \cdot 1 = nx^n + x^n = (n+1)x^n.$$

Our claim is true for $n+1$ and therefore it is true for all $n \geq 1$.

Lemma 3.6: If g is differentiable at a and $g(a) \neq 0$, then $\frac{1}{g}$ is differentiable at a, and

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{[g(a)]^2}.$$

Proof:

$$\begin{aligned}\left(\frac{1}{g}\right)'(x) &= \lim_{h\to 0} \frac{\frac{1}{g}(x+h) - \frac{1}{g}(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \lim_{h\to 0} \frac{g(x) - g(x+h)}{h \cdot g(x) \cdot g(x+h)} \\ &= \lim_{h\to 0} \left[\frac{-1}{g(x) \cdot g(x+h)}\right] \cdot \left[\frac{g(x+h) - g(x)}{h}\right] = -\frac{1}{g^2(x)} \cdot g'(x).\end{aligned}$$

3.2.2 The quotient rule

Theorem 3.7 [Quotient rule]: If f and g are differentiable at a and $g(a) \neq 0$, then $\frac{f}{g}$ is differentiable at a, and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{[g(a)]^2}.$$

Proof: `Do at home as an exercise.`

Hint: note that

$$\left(\frac{f}{g}\right)(x) = f(x) \cdot \left(\frac{1}{g}\right)(x)$$

and make use of the product rule and Lemma 3.6.

3.2.3 Examples

(1) $f(x) = \frac{x}{x^2+1}$. We have

$$f'(x) \overset{\text{QR}}{=} \frac{(x)' \cdot (x^2+1) - x \cdot (x^2+1)'}{(x^2+1)^2} = \frac{1 \cdot (x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

(2) $f(x) = \frac{1}{x}$ We have

$$f'(x) \overset{\text{QR}}{=} \frac{(1)' \cdot x - 1 \cdot (x)'}{x^2} = \frac{0-1}{x^2} = -\frac{1}{x^2}.$$

(3) $f(x) = x^{-n} = \frac{1}{x^n}$ We have

$$f'(x) \overset{\text{QR}}{=} \frac{(1)' \cdot x^n - 1 \cdot (x^n)'}{x^{2n}} = \frac{0 - 1 \cdot n \cdot x^{n-1}}{x^{2n}} = -\frac{n \cdot x^{n-1}}{x^{2n}} = -n \cdot x^{n-1-2n} = -nx^{-n-1}.$$

3.3 Derivatives of special functions

For the moment, and without proof, we shall use the following information

$$\sin'(a) = \cos(a) \quad \text{for all} \quad a,$$
$$\cos'(a) = -\sin(a) \quad \text{for all} \quad a.$$

3.3.1 Examples

(1) $f(x) = x\sin x$. We have $f'(x) = \sin x + x\cos x$.

(2) We write $\sin^k x = (\sin x)^k$ and $\cos^k x = (\cos x)^k$.
If $g(x) = \sin^2 x + \cos^2 x$, then $g'(x) = 0$ by the product rule.

$$g'(x) = 2\sin x \cdot \cos x + 2\cos x \cdot (-1) \cdot \sin x = 0.$$

Note that $\cos^2 x + \sin^2 x = 1$, so the previous result is not surprising.

3.3.2 The chain rule

We do not know yet how to differentiate functions such as $f(x) = \sin(x^3)$ and $g(x) = \cos\left(\frac{1}{3+x^2}\right)$. Notice that f is the composition of $f_2(x) = \sin x$ and $f_1(x) = x^3$, that is, $f(x) = f_2(f_1(x)) = f_2(x^3) = \sin(x^3)$.

Definition: If $\phi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ are functions, their composition $\psi \circ \phi$ is a function with domain A and range C, such that

$$\psi \circ \phi : A \longrightarrow C \quad \text{and} \quad (\psi \circ \phi)(x) = \psi(\phi(x)) \quad \text{for all} \quad x \in A.$$

The extremely important formula for the differentiation of a composition of two functions is called the chain rule.

Theorem 3.8 [Chain rule]: If g is differentiable at a and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a and $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.
Proof: Define a function φ as follows:

$$\varphi(h) = \begin{cases} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)}, & \text{if} \quad g(a+h) - g(a) \neq 0 \\ f'(g(a)), & \text{if} \quad g(a+h) - g(a) = 0 \end{cases}$$

We first show that φ is continuous at $h = 0$. Note that when $h = 0$, $\varphi(h = 0) = f'(g(a))$ as $g(a+h) - g(a) = 0$ when $h = 0$. When $h \neq 0$ and small, $g(a+h) - g(a)$ is also small, so if $g(a+h) - g(a)$ is not zero, then $\varphi(h)$ will be close to $f'(g(a))$; and if it is zero, then $\varphi(h)$ actually equals $f'(g(a)) = \varphi(h = 0)$, which is even better. We conclude that φ is continuous at $h = 0$.

We therefore have

$$\lim_{h\to 0} \varphi(h) = f'(g(a)).$$

The rest of the proof is easy. Since

$$\frac{f(g(a+h)) - f(g(a))}{h} = \varphi(h) \cdot \frac{g(a+h) - g(a)}{h}$$

holds if $g(a+h) - g(a) \neq 0$, and even if $g(a+h) - g(a) = 0$ (because in that case both sides are equal to zero), we arrive at

$$\lim_{h\to 0} \frac{f(g(a+h)) - f(g(a))}{h} = \lim_{h\to 0} \left[\varphi(h) \cdot \frac{g(a+h) - g(a)}{h}\right] = \lim_{h\to 0} \varphi(h) \cdot \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(g(a+h)) - f(g(a))}{h} = f'(g(a)) \cdot g'(a),$$

as we wanted to show.

3.3.3 Examples

(1) $f(x) = \sin(x^3)$. We have $f'(x) = 3x^2 \cos(x^3)$.
If $f_1(x) = \sin(x)$ and $f_2(x) = x^3$, we have $f(x) = (f_1 \circ f_2)(x)$. We apply the chain rule

$$f'(x) = f_1'(f_2(x)) \cdot f_2'(x) = \cos(f_2(x)) \cdot 3x^2 = \cos(x^3) \cdot 3x^2.$$

(2) $g(x) = \cos\left(\frac{1}{3+x^2}\right)$. Notice that g is the composition of $g_1(x) = \cos x$ and $g_2(x) = \frac{1}{3+x^2}$. We have $g'(x) = \frac{2x}{(3+x^2)^2} \sin\left(\frac{1}{3+x^2}\right)$. We apply the chain rule

$$g'(x) = g_1'(g_2(x)) \cdot g_2'(x) = -\sin(g_2(x)) \cdot \left(\frac{-1 \cdot 2x}{(3+x^2)^2}\right) = \sin\left(\frac{1}{3+x^2}\right) \cdot \left(\frac{2x}{(3+x^2)^2}\right).$$

(3) $h(x) = \sin^2\left(\sin^2(x)\right)$. We have $h'(x) = 2 \cdot \sin(\sin^2 x) \cdot \cos(\sin^2 x) \cdot 2 \sin x \cdot \cos x$. Note that we can write $h(x) = h_1 \circ h_1(x)$ with $h_1(x) = \sin^2(x)$. We apply the chain rule

$$h'(x) = h_1'(h_1(x)) \cdot h_1'(x) = 2\sin(h_1(x)) \cdot \cos(h_1(x)) \cdot h_1'(x)$$

$$\Rightarrow h'(x) = 2\sin(h_1(x)) \cdot \cos(h_1(x)) \cdot 2\sin x \cos x = 2\sin(\sin^2 x) \cdot \cos(\sin^2 x) \cdot 2\sin x \cos x.$$

Chapter 4

Special functions and their derivatives

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4.1 The exponential function

The Euler constant e and the associated exponential function e^x can be defined in several different ways:

$$e \approx 2.718281828459 \quad \text{and} \quad e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n.$$

The exponential function is also denoted by exp, *i.e.*, $\exp(x) = e^x$.
The exp function has the unique feature that $\exp' = \exp$. Note that $e^x > 0$.

4.1.1 The logarithmic function

The inverse of exp is the function

$$\log : (0, +\infty) \longrightarrow \mathbb{R}.$$

Thus $\log(\exp(x)) = x$ for all $x \in \mathbb{R}$ and $\exp(\log(y)) = y$ for all $y > 0$.
The derivative of log is

$$\log'(x) = \frac{1}{x} \quad \text{for all} \quad x > 0.$$

For $a > 0$ we define

$$\exp_a(x) = e^{x \log a} = \exp(x \log a).$$

We also write a^x for $\exp_a(x)$.

4.1.2 General properties of exponents

For any $a > 0$ and $b > 0$ and for any real numbers x and y
(1) $a^0 = 1$

(2) $a^x \cdot a^y = a^{x+y}$

(3) $\frac{a^x}{a^y} = a^{x-y}$

(4) $(a^x)^y = a^{x \cdot y}$

(5) $a^{-y} = \frac{1}{a^y}$

(6) $(a \cdot b)^x = a^x \cdot b^x$

(7) $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

4.1.3 Logarithmic functions

For $a > 0$ and $a \neq 1$, the inverse function of $f(x) = a^x$ is denoted by $\log_a$.

Claim: For $a > 0$ and $a \neq 1$, we have

$$\log_a(x) = \frac{\log x}{\log a} \quad \text{for all} \quad x > 0.$$

Proof:

$$\frac{\log(a^x)}{\log a} = \frac{\log\left(e^{x \cdot \log a}\right)}{\log a} = \frac{x \cdot \log a}{\log a} = x.$$

We conclude that

$$x = \frac{\log(a^x)}{\log a}.$$

We introduce $y = a^x$ so that by definition of the function $\log_a$ (the inverse function of $f(x) = a^x$) we have $x = \log_a(y)$. We can now write

$$x = \log_a(y) = \frac{\log(a^x)}{\log a} = \frac{\log(y)}{\log a} \Rightarrow \log_a(y) = \frac{\log(y)}{\log a},$$

as we wanted to show.

4.1.4 Basic properties of the logarithmic functions

For any base $a > 0$, $a \neq 1$ and for any real numbers $x, y > 0$:
(1) $\log e = 1$

(2) $\log(xy) = \log x + \log y$.

(3) $\log\left(\frac{x}{y}\right) = \log x - \log y$

(4) $\log(x^y) = y \log x$

(5) $\log_a a = 1$

(6) $\log_a(xy) = \log_a x + \log_a y$.

(7) $\log_a(x^y) = y \cdot \log_a x$

(8) $\log_a(x) = \frac{\log x}{\log a}$

(9) $\log_e = \log$.

4.1.5 Derivatives of $\exp_a$ and $\log_a$

Theorem 4.1: (a) For $a > 0$, we have

$$\exp_a'(x) = \log a \cdot \exp_a(x) = \log a \cdot a^x \quad \text{for all} \quad x.$$

In the special case $a = 1$, we have $\exp_1'(x) = 0 \quad \text{for all} \quad x$.
(b) For $a > 0$, $a \neq 1$, we have

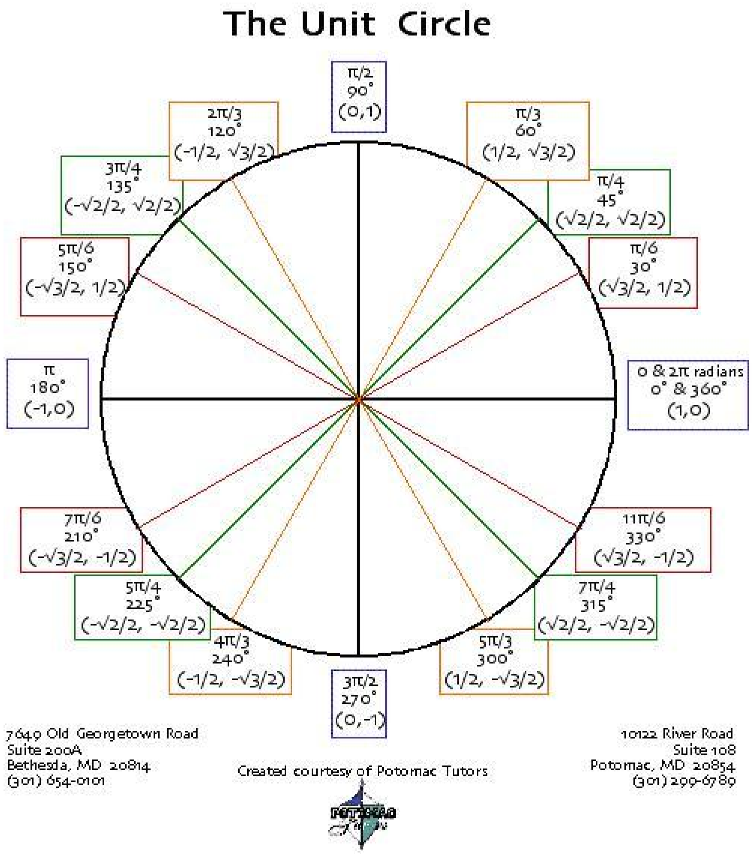
$$\log_a'(x) = \frac{1}{x \cdot \log a} \quad \text{for all} \quad x > 0.$$

Proof:

$$\text{(a)} \quad \exp_a'(x) = \left(e^{x \cdot \log a}\right)' \overset{\text{CR}}{=} e^{x \cdot \log a} \cdot \log a = \exp_a(x) \cdot \log a.$$

$$\text{(b)} \quad \log_a'(x) = \left(\frac{\log x}{\log a}\right)' = \frac{\log' x}{\log a} = \frac{1}{x \cdot \log a}.$$

4.2 Trigonometric functions



The six trigonometric functions are defined as follows

$$
\begin{aligned}
\sin\theta &= y, \\
\cos\theta &= x, \\
\tan\theta &= \frac{\sin\theta}{\cos\theta}, \quad x \neq 0 \\
\csc\theta &= \frac{1}{\sin\theta}, \quad y \neq 0 \\
\sec\theta &= \frac{1}{\cos\theta}, \quad x \neq 0 \\
\cot\theta &= \frac{\cos\theta}{\sin\theta}, \quad y \neq 0
\end{aligned}
$$

`Please revise: (i) converting degrees into radians for angles in the interval` $[0, 2\pi]$`, (ii) values of the trigonometric functions for special values and (iii) plots of trigonometric functions.`

Periodicity: The definitions of the sine and cosine functions imply that they are periodic with period 2π. That is,

$$\sin(\theta + 2\pi) = \sin\theta \quad \text{and} \quad \cos(\theta + 2\pi) = \cos\theta.$$

It follows that the secant and cosecant functions are also periodic with period 2π. It can be verified that the tangent and cotangent functions have period π.

4.3 Hyperbolic functions

The hyperbolic sine, cosine and tangent functions, written $\sinh x$, $\cosh x$ and $\tanh x$, are defined in terms of the exponential function as follows:

$$\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2}, \quad \text{(hyperbolic sine function)}\,, \\
\cosh x &= \frac{e^x + e^{-x}}{2}, \quad \text{(hyperbolic cosine function)}\,, \\
\tanh x &= \frac{\sinh x}{\cosh x}. \quad \text{(hyperbolic tangent function)}\,.
\end{aligned}$$

Note that $\cosh x \neq 0$ for all x; thus $\tanh x$ is defined for all x.

`Homework: Plot the following functions:`

1. $f_1 : \mathbb{R} \to \mathbb{R}$ such that $f_1(x) = \cosh x$.
2. $f_2 : \mathbb{R} \to \mathbb{R}$ such that $f_2(x) = \sinh x$.
3. $f_3 : \mathbb{R} \to \mathbb{R}$ such that $f_3(x) = \tanh x$.

Theorem 4.2:

$$\begin{aligned}
\sinh' x &= \cosh x \\
\cosh' x &= \sinh x \\
\tanh' x &= \frac{1}{\cosh^2 x}
\end{aligned}$$

Proof: `Do at home as an exercise.`

Chapter 5

Inverse functions and their derivatives

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5.1 Inverse functions

Definition: A function is said to be one-to-one if there are no two distinct numbers in the domain of f at which f takes on the same value: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, if f is one-to-one and x_1, x_2 are different points of the domain, then $f(x_1) \neq f(x_2)$.

Examples: (1) $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = x^3$

(2) $g : [0, +\infty) \longrightarrow [0, +\infty)$ such that $g(x) = \sqrt{x}$

Simple geometric test: The horizontal line test can be used to determine whether a function is one-to-one.
`Draw the graph on the whiteboard.`

Theorem 5.1: If f is a one-to-one function, then there is one and only one function g with domain equal to the range of f that satisfies the equation

$$f(g(x)) = x \quad \text{for all} \quad x \quad \text{in the range of} \quad f$$

Proof: The proof is straight forward. If x is in the range of f, then f must take on the value x at some number. Since f is one-to-one, there can be only one such number. We call this number $g(x)$.

The function that we have named g in the theorem is called the inverse of f and is usually denoted by the symbol f^{-1}.

Definition: Let f be a one-to-one function. The inverse function of f, denoted by f^{-1}, is the unique function with domain equal to the range of f that satisfies the equation

$$f(f^{-1}(x)) = x \quad \text{for all} \quad x \quad \text{in the range of} \quad f$$

Warning: Do not confuse the function f^{-1} with the function $\frac{1}{f}$.

5.1.1 Examples:

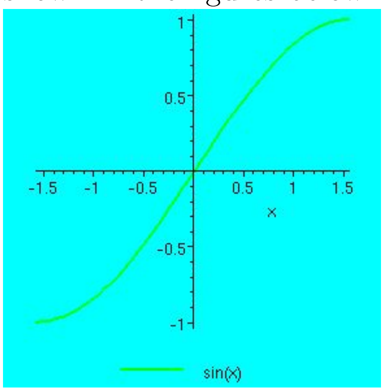
(1) $f_1 : [0, +\infty) \longrightarrow [0, +\infty)$ such that $f_1(x) = x^2$. Then, its inverse function is given by $f_1^{-1} : [0, +\infty) \longrightarrow [0, +\infty)$ with $f_1^{-1}(x) = \sqrt{x}$.

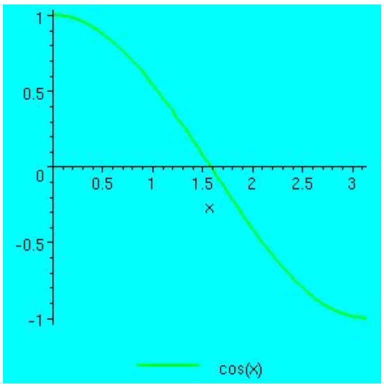
(2) $\exp^{-1} = \log$

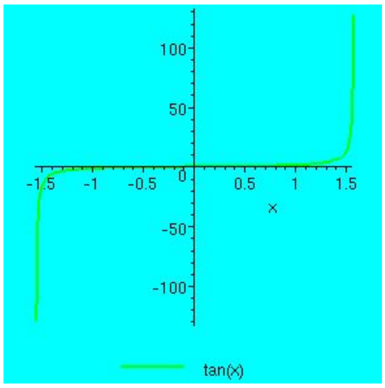
(3) $\log^{-1} = \exp$

5.2 Inverse trigonometrical functions

The trigonometrical functions sin, cos and tan are not one-to-one due to their periodicity. If their domains are suitable restricted they become one-to-one functions, as shown in the figures below.



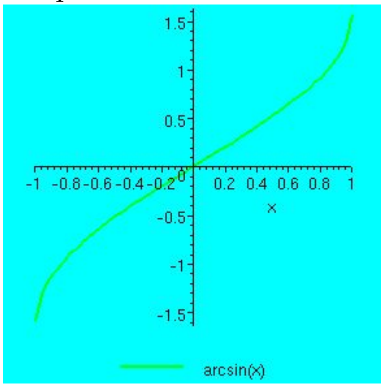


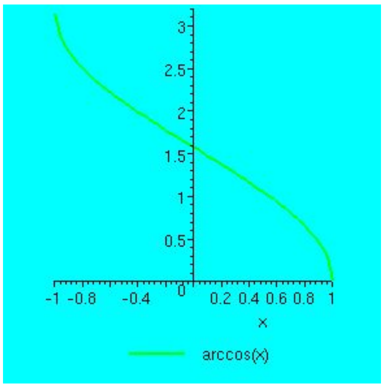


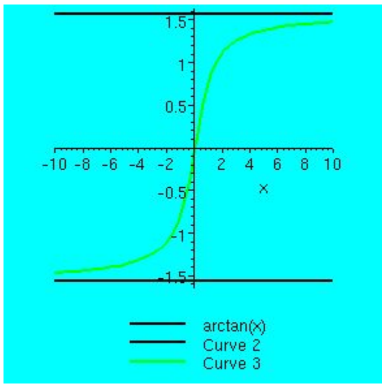
The corresponding inverse trigonometric functions which can be defined are denoted by arcsin, arccos, arctan, or, alternatively, by $\sin^{-1}$, $\cos^{-1}$ $\tan^{-1}$. These functions are defined as follows

y	$= \arcsin x$	if	$\sin y = x$;	domain:	$-1 \leq x \leq 1$	range:	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
y	$= \arccos x$	if	$\cos y = x$;	domain:	$-1 \leq x \leq 1$	range:	$0 \leq y \leq \pi$
y	$= \arctan x$	if	$\tan y = x$;	domain:	$-\infty < x < +\infty$	range:	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Graphs of these functions are shown below.







5.3 Differentiating the inverse functions

Is there a rule that allows us to express the derivative of the inverse function f^{-1} in terms of the derivative of f? The answer is positive.

Theorem 5.2: Let f be a one-to-one function defined on an interval and suppose that f is differentiable at $f^{-1}(b)$, with derivative $f'(f^{-1}(b)) \neq 0$. Then f^{-1} is differentiable at b and

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}.$$

Proof: We do not prove that f^{-1} is differentiable at b, but we shall show that the previous formula must be true if $f^{-1}(b)$ is differentiable at b.
Note that $f(f^{-1}(x)) = x$ holds for all x in the range of f. Thus, differentiating both sides of this equation we get

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = (x)' = 1,$$

where we have applied the chain rule to the left hand side. Thus, substituting b for x in the above equation

$$f'(f^{-1}(b)) \cdot (f^{-1})'(b) = 1,$$

and dividing both sides by $f'(f^{-1}(b))$ we arrive at the desired equation, namely

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}.$$

Examples: (1) Since $\exp^{-1} = \log$ we obtain

$$\log'(b) = \frac{1}{e^{\log b}} = \frac{1}{b}.$$

(2) For the inverse of trigonometric functions we get

$$\arcsin'(x) = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))},$$

when $-1 < x < 1$. Let $y = \arcsin x$. Then $\sin y = x$ and $\cos y = \sqrt{1-x^2}$, since $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $1 - \sin^2 y = \cos^2 y$. As a result,

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}},$$

whenever $-1 < x < 1$.
(3) Similarly one establishes

$$\arccos'(x) = \frac{-1}{\sqrt{1-x^2}},$$

whenever $-1 < x < 1$.
(4) Finally, we can show

$$\arctan'(x) = \frac{1}{1+x^2}.$$

Proof: For arctan we get

$$\arctan'(x) = \frac{1}{\tan'(\arctan(x))} = \cos^2(\arctan(x)),$$

because

$$\tan'(x) \overset{\text{QR}}{=} \frac{\sin'(x)\cdot\cos(x) - \sin(x)\cos'(x)}{\cos^2 x} = \frac{\cos(x)\cdot\cos(x) + \sin(x)\sin(x)}{\cos^2 x}$$
$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

Now put $y = \arctan x$, so that $x^2 = \tan^2 y = \dfrac{\sin^2 y}{\cos^2 y} = \dfrac{1-\cos^2 y}{\cos^2 y} = \dfrac{1}{\cos^2 y} - 1$, which yields

$$\cos^2 y = \frac{1}{1+x^2}.$$

Thus, we arrive at

$$\arctan'(x) = \frac{1}{1+x^2}.$$

Chapter 6

The integral

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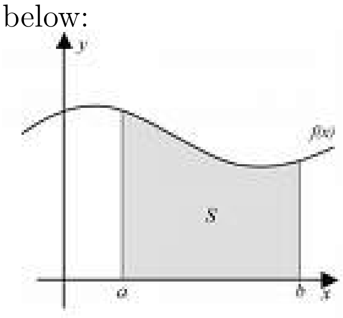
6.1 Introduction

The concept of a derivative does not display its full strength until allied with the concept of the integral.

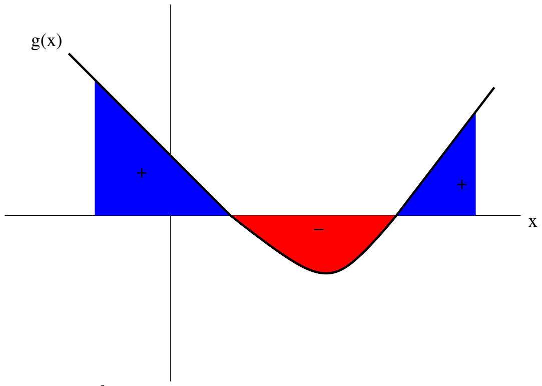
The integral, ultimately defined in quite a complicated way, formalises a simple, intuitive concept of area. Let f be a function whose graph between a and b is displayed below:



6.1.1 Interpretation

We denote the shaded region by $\mathcal{R}(f, a, b)$. The number which we will eventually assign to $\mathcal{R}(f, a, b)$ will be called the integral of f on [a,b] and denoted $\int_a^b f$ or $\int_a^b dx\ f(x)$ or $\int_a^b f(x)dx$. The quantity $\int_a^b dx f(x)$ measures the area of $\mathcal{R}(f, a, b)$.

If g is a function which also takes negative values in the interval $[a, b]$, its graph will look like this:



Here $\int_a^b dxg(x)$ will represent the difference of the area of the blue shaded region and the area of the red shaded region.

6.1.2 The speed-distance problem

Suppose that during the course of the motion the speed of a particle does not remain constant but varies continuously. How can the total distance traveled be computed then? To answer this question, we suppose that the motion begins at time a and ends at time b and that $f(t)$ is the speed at time t for $t \in [a, b]$.
`Graph`
We begin by breaking up the interval $[a, b]$ into a finite number of subintervals:

$$[t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n] \quad \text{with} \quad a = t_0 < t_1 < t_2 < \cdots < t_n = b.$$

On each subinterval $[t_{k-1}, t_k]$ the object attains a certain maximum speed M_k and a certain minimum speed m_k. If throughout the time interval $[t_{k-1}, t_k]$ the object were to move constantly at its minimum speed m_k, then it would cover a distance of

$$m_k(t_k - t_{k-1}) \quad \text{units.}$$

If instead it were to move constantly at its maximum speed M_k, then it would cover a distance of

$$M_k(t_k - t_{k-1}) \quad \text{units.}$$

As it is, the actual distance travelled on each subinterval $[t_{k-1}, t_k]$, call it s_k must lie somewhere in between; namely, we must have

$$m_k(t_k - t_{k-1}) \leq s_k \leq M_k(t_k - t_{k-1}).$$

The total distance travelled during the time interval $[a, b]$, call it s, must be the sum of the distances travelled during the subintervals $[t_{k-1}, t_k]$. In other words, we must have

$$s = s_1 + s_2 + \cdots + s_n.$$

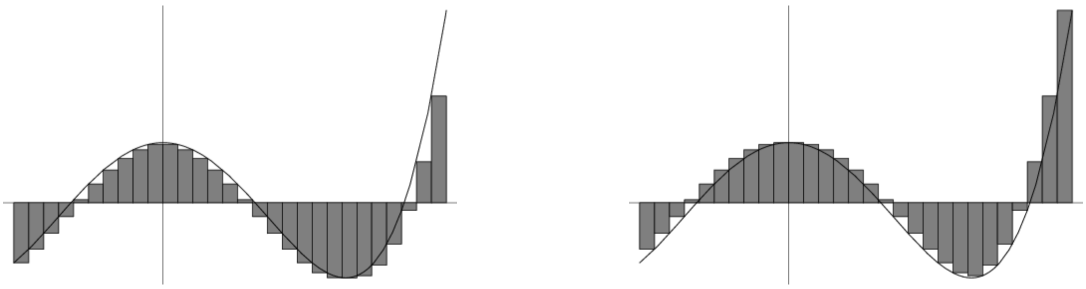


Figure 6.1: Lower sum (left figure) and Upper sum (right figure)

It follows by the addition of the inequalities that

$$\sum_{k=1}^{n} m_k(t_k - t_{k-1}) \leq \sum_{k=1}^{n} s_k = s \leq \sum_{k=1}^{n} M_k(t_k - t_{k-1}).$$

$$\sum_{k=1}^{n} m_k(t_k - t_{k-1})$$

is called a lower sum for the speed function f and

$$\sum_{k=1}^{n} M_k(t_k - t_{k-1})$$

is called an upper sum for the speed function f.

The actual distance travelled corresponds to the area of the region $\mathcal{R}(f, a, b)$, *i.e.*, $s = \int_a^b f$. Thence

$$\sum_{k=1}^{n} m_k(t_k - t_{k-1}) \leq \int_a^b dx\ f(x) \leq \sum_{k=1}^{n} M_k(t_k - t_{k-1}).$$

6.2 The definite integral

Definition: Let a, b. A partition of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a and one of which is b.

The points can be numbered so that $a = t_0 < t_1 < t_2 < \cdots < t_n = b$; we shall always assume that such a numbering has been assigned.

Definition: A function f is bounded on $[a, b]$ if there exists a positive integer N such that

$$|f(x)| < N \qquad \text{for all} \qquad x \in [a, b].$$

Definition: Suppose f is bounded on $[a, b]$ and $P = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$ is a partition of $[a, b]$. Let m_k be the minimum value of f on $[t_{k-1}, t_k]$, *i.e.*, $m_k = inf\{f(x) : x \in [t_{k-1}, t_k]\}$, and M_k be the maximum value of f on $[t_{k-1}, t_k]$, *i.e.*, $M_k = sup\{f(x) : x \in [t_{k-1}, t_k]\}$.

The lower sum of f for P, denoted by $L(f, P)$ is defined as

$$L(f, P) = \sum_{k=1}^{n} m_k(t_k - t_{k-1}).$$

The upper sum of f for P, denoted by $U(f, P)$ is defined as

$$U(f, P) = \sum_{k=1}^{n} M_k(t_k - t_{k-1}).$$

Definition (the definite integral): A function f defined on an interval $[a, b]$ which is bounded on $[a, b]$ is integrable on $[a, b]$ if there is one and only one number I that satisfies the inequality

$$L(f, P) \leq I \leq U(f, P),$$

for all partitions P of $[a, b]$.

This unique number I is called the definite integral (or more simply the integral) of f from a to b and is denoted by

$$\int_a^b f \quad \text{or} \int_a^b dx\ f(x).$$

6.2.1 Examples:

(1) Let $f(x) = x^2$. We have

$$\int_0^b dx\ f(x) = \frac{b^3}{3}.$$

Proof: Let $b > 0$ and $P_n = \{0 = t_0 < t_1 < t_2 < \cdots < t_n = b\}$ be a partition of $[0, b]$. Set $\Delta_i = t_i - t_{i-1}$, $i = 1, \ldots, n$.
The function $f(x) = x^2$ is an increasing function and this implies that $m_i = f(t_{i-1}) = t_{i-1}^2$ and $M_i = f(t_i) = t_i^2$. We then have

$$L(f, P_n) = \sum_{i=1}^{n} t_{i-1}^2(t_i - t_{i-1}) = \sum_{i=1}^{n} t_{i-1}^2 \Delta_i \qquad \text{and} \qquad U(f, P_n) = \sum_{i=1}^{n} t_i^2(t_i - t_{i-1}) = \sum_{i=1}^{n} t_i^2 \Delta_i.$$

Now suppose P_n partitions $[0, b]$ into n equal parts. Then

$$t_i = \frac{i \cdot b}{n}$$

and the lower and upper sums become

$$L(f,P_n) = \sum_{i=1}^{n} t_{i-1}^2(t_i - t_{i-1}) = \sum_{i=1}^{n}(i-1)^2\frac{b^2}{n^2}\cdot\frac{b}{n} = \sum_{i=1}^{n}(i-1)^2\frac{b^3}{n^3} = \frac{b^3}{n^3}\sum_{i=1}^{n}(i-1)^2$$

$$U(f,P_n) = \sum_{i=1}^{n} t_i^2(t_i - t_{i-1}) = \sum_{i=1}^{n}(i)^2\frac{b^2}{n^2}\cdot\frac{b}{n} = \sum_{i=1}^{n}(i)^2\frac{b^3}{n^3} = \frac{b^3}{n^3}\sum_{i=1}^{n}(i)^2.$$

Using the formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

we get

$$\sum_{i=1}^{n}(i)^2 = \frac{1}{6}n(n+1)(2n+1),$$

and

$$\sum_{i=1}^{n}(i-1)^2 = 0^2+2^2+3^2+\cdots+(n-1)^2 = \sum_{i=0}^{n-1}(i)^2 = \frac{1}{6}(n-1)(n)(2(n-1)+1) = \frac{1}{6}(n-1)(n)(2n-1),$$

so that

$$L(f,P_n) = \frac{b^3}{n^3}\cdot\frac{1}{6}(n-1)(n)(2n-1) \qquad \text{and} \qquad U(f,P_n) = \frac{b^3}{n^3}\cdot\frac{1}{6}(n)(n+1)(2n+1).$$

It is not hard to show that (make use of Mathematical Induction)

$$L(f,P_n) \le \frac{b^3}{3} \le U(f,P_n)$$

and that

$$U(f,P_n) - L(f,P_n) = \frac{b^3}{n}$$

can be made as small as desired by choosing n sufficiently large.

This sort of reasoning then shows that

$$\int_0^b dx\ f(x) = \frac{b^3}{3}.$$

(2) Let $g(x) = x$. We have

$$\int_0^b dx\ g(x) = \frac{b^2}{2}.$$

Proof: Let $b > 0$ and $P_n = \{0 = t_0 < t_1 < t_2 < \cdots < t_n = b\}$ be a partition of $[0,b]$. Set $\Delta_i = t_i - t_{i-1}$, $i = 1,\ldots,n$.

The function $g(x) = x$ is an increasing function and this implies that $m_i = g(t_{i-1}) = t_{i-1}$ and $M_i = g(t_i) = t_i$. We then have

$$L(g, P_n) = \sum_{i=1}^{n} t_{i-1}(t_i - t_{i-1}) = \sum_{i=1}^{n} t_{i-1}\Delta_i \qquad \text{and} \qquad U(g, P_n) = \sum_{i=1}^{n} t_i(t_i - t_{i-1}) = \sum_{i=1}^{n} t_i\Delta_i.$$

Now suppose P_n partitions $[0, b]$ into n equal parts. Then

$$t_i = \frac{i \cdot b}{n}$$

and the lower and upper sums become

$$L(g, P_n) = \sum_{i=1}^{n} t_{i-1}(t_i - t_{i-1}) = \sum_{i=1}^{n} (i-1)\frac{b^2}{n^2} = \frac{b^2}{n^2}\sum_{i=1}^{n}(i-1)$$

$$U(g, P_n) = \sum_{i=1}^{n} t_i(t_i - t_{i-1}) = \sum_{i=1}^{n} (i)\frac{b^2}{n^2} = \frac{b^2}{n^2}\sum_{i=1}^{n}(i).$$

Using the formula

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$$

we get

$$\sum_{i=1}^{n}(i) = \frac{1}{2}n(n+1),$$

and

$$\sum_{i=1}^{n}(i-1) = 0 + 2 + 3 + \cdots + (n-1) = \sum_{i=0}^{n-1}(i) = \frac{1}{2}(n-1)(n),$$

so that

$$L(g, P_n) = \frac{b^2}{n^2}\cdot\frac{1}{2}(n-1)(n) = \frac{(n-1)}{n}\cdot\frac{b^2}{2} \qquad \text{and} \qquad U(g, P_n) = \frac{b^2}{n^2}\cdot\frac{1}{2}(n)(n+1) = \frac{(n+1)}{n}\cdot\frac{b^2}{2}.$$

It is not hard to show that (make use of Mathematical Induction)

$$\frac{(n-1)}{n}\cdot\frac{b^2}{2} \le \frac{b^2}{2} \le \frac{(n+1)}{n}\cdot\frac{b^2}{2},$$

we get

$$L(g, P_n) \le \frac{b^2}{2} \le U(g, P_n)$$

and that

$$U(g, P_n) - L(g, P_n) = \frac{2}{n}\cdot\frac{b^2}{2} = \frac{b^2}{n}$$

can be made as small as desired by choosing n sufficiently large.

This sort of reasoning then shows that

$$\int_0^b dx\ g(x) = \frac{b^2}{2}.$$

(3) Here is an example of a function that is not integrable. Define

$$h(x) = \begin{cases} 0 & \text{if} \quad x \quad \text{is rational} \\ 1 & \text{if} \quad x \quad \text{is irrational} \end{cases} \tag{6.1}$$

Proof: If we take a partition $P_n = \{0 = t_0 < t_1 < t_2 < \cdots < t_n = b\}$ of $[a, b]$, we get

$$L(h, P_n) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = 0 \qquad \text{and} \qquad U(f, P_n) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) = b - a,$$

because the minimum value of the function h on any interval is 0 and the maximum value of h on any interval is 1.

The two examples of computing

$$\int_0^b dx\ f(x)$$

show that this can be quite laborious a task.
In the next section we shall be introducing a powerful tool for calculating integrals, the so-called <u>fundamental theorem of calculus</u>, which connects differentiation and integration. We finish this section mentioning this important result.

Theorem 6.1: The first fundamental theorem of calculus
Suppose f is continuous on the interval $[a, b]$. Then f is integrable on $[a, b]$.
Proof: It will be given in the following section. □

6.3 The fundamental theorem of calculus

Theorem 6.1: (The first fundamental theorem of calculus)
Let f be integrable on $[a, b]$ and define F on $[a, b]$ by

$$F(t) = \int_a^t dx\ f(x).$$

If f is continuous at c in the interval $[a, b]$, then F is differentiable at c and $F'(c) = f(c)$.
Proof: Let $h > 0$. Then

$$\frac{F(c+h) - F(c)}{h} = \frac{1}{h}\left[\int_a^{c+h} dx\ f(x) - \int_a^c dx\ f(x)\right] = \frac{1}{h}\int_c^{c+h} dx\ f(x).$$

Now define

$m_h =$ minimum value of f on $[c, c + h]$;

$M_h =$ maximum value of f on $[c, c+h]$;

By definition of the integral we have

$$h \cdot m_h \leq \int_c^{c+h} dx\ f(x) \leq h \cdot M_h.$$

Therefore

$$m_h \leq \frac{1}{h} \int_c^{c+h} dx\ f(x) \leq M_h.$$

Since f is continuous at c we have

$$\lim_{h \to 0} m_h = \lim_{h \to 0} M_h = f(c).$$

Thus,

$$F'(c) = \lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = \lim_{h \to 0} \left[\frac{1}{h} \int_c^{c+h} dx\ f(x) \right] = f(c).$$

[If $h < 0$, only a few details of the argument have to be changed.] □

Lemma 6.2: Let g, h be differentiable functions on $[a, b]$. If $g'(x) = h'(x)$ for all $x \in (a, b)$, then there exists a constant C such that $g(x) = h(x) + c$ for all $x \in (a, b)$.

Proof: Set $\varphi(x) = g(x) - h(x)$. Then $\varphi'(x) = 0$ for all $x \in [a, b]$. This implies that φ is constant on $[a, b]$. (This should be intuitively clear; we cannot give a rigorous proof at this stage.) Thus, there is a constant C such that $\varphi(x) = C$ for all $c \in [a, b]$ and hence $g(x) = h(x) + C$. □

Corollary 6.3: If f is continuous on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b dx\ f(x) = g(b) - g(a).$$

Proof: Let

$$F(t) = \int_a^t dx\ f(x).$$

Then $F' = f$ by Theorem 6.1. Thus $F' = g'$. So it follows by Lemma 6.2 that there is a constant C such that $F = g + C$. Now $F(a) = 0$ and

$$F(b) = \int_a^b dx\ f(x).$$

Hence

$$\int_a^b dx\ f(x) = F(b) - F(a) = (g(b) + C) - (g(a) + C) = g(b) - g(a).$$

Since $F(a) = 0$ and $F(a) = g(a) + C$, we get $C = -g(a)$. □
The next theorem strengthens Theorem 6.1.

Theorem 6.4: (The second fundamental theorem of calculus)
If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b dx\ f(x) = g(b) - g(a).$$

Proof: Not given. □

6.3.1 Examples

1. $f(x) = x^n$ for some $n \geq 1$. Then $g'(x) = f(x)$, where

$$g(x) = \frac{x^{n+1}}{n+1}.$$

   Hence

$$\int_a^b dx\ f(x) = g(b) - g(a) = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}.$$

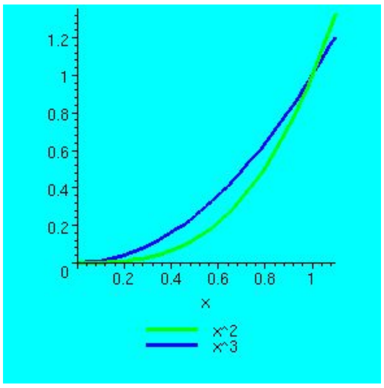
2. If $n > 1$ and $f(x) = x^{-n}$ and $0 < a < b$, then $g'(x) = f(x)$, where

$$g(x) = \frac{x^{-n+1}}{-n+1}.$$

   Hence

$$\int_a^b dx\ f(x) = g(b) - g(a) = \frac{b^{-n+1}}{-n+1} - \frac{a^{-n+1}}{-n+1}.$$

3. Find the area of the region between the graphs of the functions $f(x) = x^2$ and $g(x) = x^3$ on the interval $[0, 1]$.



   If $0 \leq x \leq 1$, then $0 \leq x^3 \leq x^2$, so that the graph of g lies below that of f. The area of the region of interest to us is therefore

$$\mathcal{R}(f, 0, 1) - \mathcal{R}(g, 0, 1),$$

   which is

$$\int_0^1 dx\ x^2 - \int_0^1 dx\ x^3 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

6.4 Integration by substitution

Every computation of a derivative yields, according to the Fundamental Theorem of Calculus, a formula about integrals. For example, if

$$F(x) = x(\log x) - x,$$

then $F'(x) = \log x$. Consequently,

$$\int_a^b dx\ \log x = F(b) - F(a) = b(\log b) - b - [a(\log a) - a],$$

when $0 < a < b$.
Formulas of this sort are simplified considerably if we adopt the notation

$$F(x)\Big|_a^b = F(b) - F(a).$$

We may then write

$$\int_a^b dx\ \log x = F(b) - F(a) = x(\log x) - x\Big|_a^b.$$

Definition: A function F satisfying $F' = f$ is called a primitive of f.
Notice that a continuous function f always has a primitive, namely

$$F(x) = \int_a^x dx\ f(x).$$

In this section we will try to find a primitive which can be written in terms of familiar functions. A function which can be written in this way is called an elementary function. To be precise, an elementary function is one which can be obtained by addition, multiplication, division and composition from the rational functions, the trigonometric functions and their inverses and the functions log and exp.

It should be stated at the very outset that elementary primitives usually cannot be found. For example, there is no elementary function F such that

$$F'(x) = e^{-x^2} \qquad \text{for all} \qquad x.$$

Every rule for the computation of a derivative gives a formula about integrals. **In particular, the product rule gives rise to Integration by Parts.** We use the acronym FTC for Fundamental Theorem of Calculus.

Theorem 6.5 [Integration by parts]: If f' and g' are continuous on $[a,b]$, then

$$\int_a^b dx\ f(x)g'(x) = f(x)g(x)\Big|_a^b - \int_a^b dx\ f'(x)g(x).$$

Proof: Let $F = f \cdot g$. By the Product Rule for differentiation we have

$$F' = f' \cdot g + f \cdot g'.$$

Hence, by the FTC,

$$\int_a^b dx\ [f'(x) \cdot g(x) + f(x)g'(x)] = \int_a^b dx\ f'(x) \cdot g(x) + \int_a^b dx\ f(x)g'(x) = f(b)g(b) - f(a)g(a).$$

Hence

$$\int_a^b dx\ f(x)g'(x) = f(x)g(x)\Big|_a^b - \int_a^b dx\ f'(x)g(x).$$

6.4.1 Examples

1. Example 1

   $$\int_a^b dx\ x\ e^x = x\ e^x\Big|_a^b - \int_a^b dx\ 1 \cdot e^x = (x-1) \cdot e^x\Big|_a^b,$$

   where we have chosen $f = x$ and $g' = e^x$, so that $f' = 1$ and $g = e^x$.

2. Example 2

   $$\int_a^b dx\ x\ \sin x = x(-\cos x)\Big|_a^b - \int_a^b dx\ 1 \cdot (-\cos x). = x(-\cos x)\Big|_a^b + \sin x\Big|_a^b,$$

   where we have chosen $f = x$ and $g' = \sin x$, so that $f' = 1$ and $g = -\cos x$.

3. Example 3: There is a special trick which often works with integration by parts, namely to consider the function g' to be the factor 1, which can always be written in.

   $$\int_a^b dx\ \log x = \int_a^b dx\ 1 \cdot \log x = x \log x\Big|_a^b - \int_a^b dx\ x \cdot \frac{1}{x}. = x \log x\Big|_a^b - x\Big|_a^b = x(\log x - 1)\Big|_a^b$$

   where we have chosen $f = \log x$ and $g' = 1$, so that $f' = 1/x$ and $g = x$.

4. Example 4: Another trick is to use integration by parts to find $\int h$ in terms of h again, and then solve for $\int h$.

   $$\int_a^b dx\ \frac{1}{x} \log x = \log x \log x\Big|_a^b - \int_a^b dx\ \frac{1}{x} \cdot \log x,$$

   where we have chosen $f = \log x$ and $g' = 1/x$, so that $f' = 1/x$ and $g = \log x$. This implies that

   $$2\int_a^b dx\ \frac{1}{x} \log x = \log x \log x\Big|_a^b = (\log x)^2\Big|_a^b \Rightarrow \int_a^b dx\ \frac{1}{x} \log x = \frac{1}{2}(\log x)^2\Big|_a^b.$$

6.5 The indefinite integral

Definition: If f is a continuous function, then it has a primitive $F(x) = \int_a^x dx\ f(x)$.

We also know that if G is another primitive of f, then F and G differ by a constant C, *i.e.*,

$$F(x) = G(x) + C.$$

The symbol $\int f$ or $\int dx f(x)$ (without boundaries) is used to denote a "primitive of f" or, more precisely, "the collection of all primitives of f".

A formula like

$$\int dx\ x^3 = \frac{x^4}{4},$$

means that the function $F(x) = \frac{x^4}{4}$ satisfies $F'(x) = x^3$. Some people write

$$\int dx\ x^3 = \frac{x^4}{4} + C,$$

to emphasize that the primitives of $f(x) = x^3$ are precisely the functions of the form $F(x) = \frac{x^4}{4} + C$ for some real number C. We shall omit C as the concern for this constant is merely an annoyance.

A function $\int dx\ f(x)$, *i.e.*, a primitive of f, is often called an indefinite integral of f, while $\int_a^b dx\ f(x)$ is called, by way of contrast, a definite integral.

Theorem 6.6 [Integration by parts, indefinite form]: If f' and g' are continuous on $[a, b]$, then

$$\int dx\ f(x)g'(x) = f(x)g(x) - \int dx\ f'(x)g(x).$$

Proof: Let G be a primitive of $f'g$. Put

$$F(x) = f(x) \cdot g(x) - G(x).$$

Then

$$F'(x) \stackrel{\text{PR}}{=} f'(x) \cdot g(x) + f(x) \cdot g'(x) - G'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) - f'(x) \cdot g(x) = f(x) \cdot g'(x). \square$$

6.5.1 Examples

1. Example 1

$$\int dx\ e^x\ \sin x = e^x \cdot (-\cos x) - \int dx\ e^x \cdot (-\cos x) = e^x \cdot (-\cos x) + \int dx\ e^x \cdot (\cos x),$$

where we have chosen $f = e^x$ and $g' = \sin x$, so that $f' = e^x$ and $g = -\cos x$. We integrate by parts again, with the choice $u = e^x$ and $v' = \cos x$, so that $u' = e^x$ and $v = \sin x$, so that

$$\int dx\ e^x\ \sin x = e^x \cdot (-\cos x) + [e^x \cdot \sin x - \int dx\ e^x \cdot (\sin x)],$$

Therefore

$$2\int dx\ e^x\ \sin x = e^x \cdot (-\cos x) + e^x \cdot \sin x = e^x(-\cos x + \sin x)$$

$$\Rightarrow \int dx\ e^x\ \sin x = \frac{1}{2}e^x(-\cos x + \sin x).$$

2. Example 2

$$\int dx\ (\log x)^2 = \int dx\ (\log x) \cdot (\log x),$$

where we have chosen $f = \log x$ and $g' = \log x$. We make use of our previous result (see previous example 3) to obtain $f' = 1/x$ and $g = [x(\log x) - x]$:

$$\begin{aligned}\int dx\ (\log x)^2 &= \log x[x(\log x) - x] - \int dx\ \frac{1}{x}[x(\log x) - x] \\ &= \log x[x(\log x) - x] - \int dx\ [(\log x) - 1] \\ &= \log x[x(\log x) - x] - \int dx\ \log x + \int dx\ 1 \\ &= \log x[x(\log x) - x] - [x(\log x) - x] + x = x(\log x)^2 - 2x(\log x) + 2x.\end{aligned}$$

3. Example 3 We choose $f = x^2$ and $g' = e^x$ so that $f' = 2x$ and $g = e^x$:

$$\begin{aligned}\int dx\ x^2 \cdot e^x &= x^2 \cdot e^x - \int dx\ 2x \cdot e^x \\ &= x^2 \cdot e^x - 2\int dx\ x \cdot e^x \qquad \text{integration by parts} \qquad u = x, v' = e^x \\ &= x^2 \cdot e^x - 2[x \cdot e^x - \int dx\ 1 \cdot e^x] \\ &= x^2 \cdot e^x - 2[x \cdot e^x - e^x] = e^x[x^2 - 2x + 2]\end{aligned}$$

6.5.2 The substitution formula

Theorem 6.7 [Substitution formula]: If f and g' are continuous, then

$$\int_{g(a)}^{g(b)} du\ f(u) = \int_a^b dx\ f(g(x)) \cdot g'(x).$$

Proof: If F is a primitive of f, then the left side is $F(g(b)) - F(g(a))$. On the other hand,

$$(F \circ g)' \overset{\text{CR}}{=} (F' \circ g) \cdot g' = (f \circ g) \cdot g',$$

by the Chain Rule.

So $F \circ g$ is a primitive of $(f \circ g) \cdot g'$ and the right side is $(F \circ g)(b) - (F \circ g)(a)$.

The simplest uses of the substitution formula depend upon recognising that a given function is of the form $(f \circ g) \cdot g'$. Let SF stand for substitution formula

6.5.3 Examples

1. Example 1

$$\int_a^b dx\ \sin^5 x \cdot \cos x.$$

Consider $g(x) = \sin x$ and $f(u) = u^5$, so that $g'(x) = \cos x$ and $(f \circ g)(x) = \sin^5 x$.

$$\int_a^b dx\ \sin^5 x \cdot \cos x \ \stackrel{\text{SF}}{=}\ \int_{g(a)}^{g(b)} du\ f(u) = \int_{\sin(a)}^{\sin(b)} du\ u^5 = \frac{\sin^6 b}{6} - \frac{\sin^6 a}{6}.$$

2. Example 2

$$\int_a^b dx\ \tan x.$$

Consider $g(x) = \cos x$ and $f(u) = \frac{1}{u}$.

$$\begin{aligned}\int_a^b dx\ \tan x &= -\int_a^b dx\ \frac{-\sin x}{\cos x} = -\int_a^b dx\ f(g(x)) \cdot g'(x)\\ &\stackrel{\text{SF}}{=} -\int_{g(a)}^{g(b)} du\ f(u) = -\int_{\cos(a)}^{\cos(b)} du\ \frac{1}{u} = \log(\cos a) - \log(\cos b).\end{aligned}$$

3. Example 3

$$\int_a^b dx\ \frac{1}{x \log x}.$$

Consider $g(x) = \log x$ and $f(u) = \frac{1}{u}$.

$$\begin{aligned}\int_a^b dx\ \frac{1}{x \log x} &= \int_a^b dx\ f(g(x)) \cdot g'(x) \stackrel{\text{SF}}{=} \int_{g(a)}^{g(b)} du\ f(u)\\ &= \int_{\log(a)}^{\log(b)} du\ \frac{1}{u} = \log(\log b) - \log(\log a).\end{aligned}$$

The uses of the substitution formula can be shortened by eliminating the intermediate steps, which involve writing

$$\int_a^b dx\ f(g(x)) \cdot g'(x) = \int_{g(a)}^{g(b)} du\ f(u),$$

by noticing the following: To go from the left side to the right side, substitute
(1) u for $g(x)$
(2) du for $dxg'(x)$
and change the limits of integration.
Our first example then becomes

1. Example 1
$$\int_a^b dx\ \sin^5 x \cdot \cos x.$$
Substitute u for $\sin x$ and $du = dx \cos x$ so that
$$\int_a^b dx\ \sin^5 x \cdot \cos x = \int_{\sin(a)}^{\sin(b)} du\ u^5 = \frac{\sin^6 b}{6} - \frac{\sin^6 a}{6}.$$
2. Example 2
$$\int_a^b dx\ \tan x.$$
Substitute u for $\cos x$ and $du = -dx \sin x$ so that
$$\int_a^b dx\ \tan x = -\int_a^b dx\ \frac{-\sin x}{\cos x} = -\int_{\cos(a)}^{\cos(b)} du\ \frac{1}{u} = \log(\cos a) - \log(\cos b).$$
3. Example 3
$$\int_a^b dx\ \frac{1}{x \log x}.$$
Substitute u for $\log x$ and $du = dx(1/x)$ so that
$$\int_a^b dx\ \frac{1}{x \log x} = \int_{\log(a)}^{\log(b)} du\ \frac{1}{u} = \log(\log b) - \log(\log a).$$

Next, we shall be interested in primitives rather than definite integrals.
If we can find
$$\int_a^b dx\ f(x)$$
for all a and b, then we can certainly find
$$\int dx\ f(x).$$
For example, since
$$\int_a^b dx\ \sin^5 x \cdot \cos x = \frac{\sin^6 b}{6} - \frac{\sin^6 a}{6},$$
it follows that
$$\int dx\ \sin^5 x \cdot \cos x = \frac{\sin^6 x}{x}.$$
Similarly
$$\int dx\ \tan x = -\log(\cos x).$$

It is uneconomical to obtain primitives from the substitution formula by first finding definite integrals. Instead, the two steps can be combined to yield the following procedure:
(1) Let u for $g(x)$, so that du for $dxg'(x)$.
(2) Find a primitive (as an expression involving u).
(3) Substitute $g(x)$ back for u.

6.5.4 Examples

1. Example 1

$$\int dx\ \sin^5 x \cdot \cos x.$$

(1) Substitute u for $\sin x$ and $du = dx \cos x$ so that

$$\int dx\ \sin^5 x \cdot \cos x = \int du\ u^5.$$

(2) Evaluate the previous integral

$$\int du\ u^5 = \frac{u^6}{6}.$$

(3) Remember to substitute back in terms of x, not u (u for $\sin x$)

$$\int dx\ \sin^5 x \cdot \cos x = \int du\ u^5 = \frac{u^6}{6} = \frac{\sin^6 x}{6}.$$

2. Example 2

$$\int dx\ \frac{1}{x \log x}.$$

(1) Substitute u for $\log x$ and $du = dx(1/x)$ so that

$$\int dx\ \frac{1}{x \log x} = \int du\ \frac{1}{u}.$$

(2) Evaluate the previous integral

$$\int du\ \frac{1}{u} = \log u.$$

(3) Remember to substitute back in terms of x, not u (u for $\log x$)

$$\int dx\ \frac{1}{x \log x} = \int du\ \frac{1}{u} = \log u = \log(\log x).$$

3. Example 3

$$\int dx\ \frac{x}{1+x^2}.$$

(1) Substitute u for $1+x^2$ and $du = dx 2x$ so that

$$\int dx\ \frac{x}{1+x^2} = \frac{1}{2} \int du\ \frac{1}{u}.$$

(2) Evaluate the previous integral

$$\frac{1}{2}\int du\ \frac{1}{u} = \frac{1}{2}\log u.$$

(3) Remember to substitute back in terms of x, not u (u for $1+x^2$)

$$\int dx\ \frac{x}{1+x^2}.\frac{1}{2}\int du\ \frac{1}{u} = \frac{1}{2}\log u = \frac{1}{2}\log(1+x^2).$$

4. Example 4

$$\int dx\ \arctan x.$$

Since $\arctan'(x) = \frac{1}{1+x^2}$, integration by parts yields

$$\int dx\ \arctan x = \int dx\ 1{\cdot}\arctan x = x{\cdot}\arctan x - \int dx\ x{\cdot}\frac{1}{1+x^2} = x{\cdot}\arctan x - \frac{1}{2}\log(1+x^2).$$

5. Example 5: More interesting uses of the substitution formula occur when the factor $g'(x)$ does NOT appear. Consider

$$\int dx\ \frac{1+e^x}{1-e^x}.$$

We can write

$$\int dx\ \frac{1+e^x}{1-e^x} = \int dx\ \frac{1+e^x}{1-e^x}\frac{1}{e^x}\cdot e^x.$$

We let $u = e^x$ and $du = dx\ e^x$, so that we obtain

$$\int du\ \frac{1+u}{1-u}\frac{1}{u} = \int du\ \left[\frac{2}{1-u} + \frac{1}{u}\right] = -2\log(1-u) + \log(u).$$

We substitute back to obtain

$$\int dx\ \frac{1+e^x}{1-e^x} = -2\log(1-e^x) + \log(e^x) = -2\log(1-e^x) + x.$$

Chapter 7

Complex numbers

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7.1 Introduction

There are equations that have no real solution

$$x^2 + 1 = 0,$$

since $x^2 > 0$ for all reals x. So we introduce the complex numbers.
That is the end of the process since any polynomial equation

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$$

with complex coefficients, $a_0, a_1, a_2, \cdots, a_n$, always has a complex solution, and all its solutions are complex (counting real numbers as a special kind of complex numbers). So there is no need to introduce new numbers. This is called the **Fundamental Theorem of Algebra,** although all proofs of this theorem need some Analysis.

Definition: A complex number is an ordered pair of real numbers; if $z = (a, b)$ is a complex number, then a is called the real part (notated $\mathrm{Re}(z)$) of z and b is called the imaginary part of z (notated $\mathrm{Im}(z)$). The set of complex numbers is denoted by $\mathbb{C}$. If (a, b) and (c, d) are two complex numbers we define

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d),\\ (a, b) \cdot (c, d) &= (a \cdot c - b \cdot d, a \cdot d + b \cdot c).\end{aligned}$$

Note that the + and $\cdot$ appearing on the left hand side are new symbols being defined, while the $+, -$ and $\cdot$ appearing on the right hand side are the familiar addition, subtraction and multiplication for real numbers.

Example:

$$\begin{aligned}(1,2)+\left(-\frac{1}{2},\pi\right) &= \left(\frac{1}{2},2+\pi\right)\\ (1,2)\cdot\left(-\frac{1}{2},\pi\right) &= \left(-\frac{1}{2}-2\pi,\pi-1\right).\end{aligned}$$

In particular,

$$\begin{aligned}(a,0)+(b,0) &= (a+b,0),\\ (a,0)\cdot(b,0) &= (a\cdot b,0).\end{aligned}$$

This shows that the complex numbers of the form $(a,0)$ behave precisely the same with respect to addition and multiplication of complex numbers as real numbers do with their own addition and multiplication. For this reason we will identify a real number a with the complex number $(a,0)$ and adopt the convention that $(a,0)$ will be denoted simply by a.
The familiar $a+i\cdot b$ notation for complex numbers can now be recovered if one more definition is made.

Definition: $i=(0,1)$.
Notice that $i^2=(0,1)\cdot(0,1)=(-1,0)=-1$.
Moreover,

$$(a,b)=(a,0)+(0,b)=(a,0)+(b,0)\cdot(0,1)=a+i\cdot b.$$

We want to show that the usual laws of algebra operate in $\mathbb{C}$.

If $z=a+ib$ with a,b real numbers, $a+ib$ is called the Cartesian form of z.

Definition: If $z=a+ib$ is a complex number with a,b real numbers, then the conjugate or complex conjugate $\bar{z}$ (or z^*) of z is defined as

$$\bar{z}=a-ib$$

and the absolute value or modulus $|z|$ of z is defined as

$$|z|=\sqrt{a^2+b^2}.$$

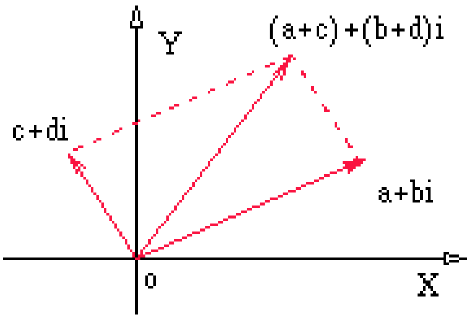
Note that
(i) $a^2+b^2\geq 0$ so that $\sqrt{a^2+b^2}$ is defined unambiguously; it denotes the non-negative real square root of a^2+b^2;
(ii) $z\cdot\bar{z}=a^2+b^2$, thus $z\cdot\bar{z}$ is a real number and $|z|=\sqrt{z\cdot\bar{z}}$, or equivalently, $|z|^2=z\cdot\bar{z}$.

7.2 The geometry of complex numbers

The complex number $z = a + ib$ can be represented graphically either as a point or as a directed line (vector) in what is called the complex plane (also the z-plane) or an Argand diagram.

The complex number $z = a+ib$ is represented either as the point with coordinates (a, b) or as the directed line from the origin to the point (a, b) with the direction directed by an arrow on the line pointing away from the origin.

Geometrical representation of the sum: The geometrical representation of the sum $z_1 + z_2$ with $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ is shown.



The rule for addition is called the triangle law for addition.
Addition is a commutative operation

$$z_1 + z_2 = z_2 + z_1,$$

so the sum may be evaluated in either of the ways.
This leads to the parallelogram law for addition.

Geometrical representation of the difference: The difference is formed in similar fashion by writing

$$z_1 - z_2 = z_1 + (-z_2)$$

and adding to z_1 the complex number $-z_2$.

Geometrical representation of the complex conjugate: The complex conjugate $\bar{z} = a - ib$ of the complex number $z = a + ib$ is the reflection of the point (a, b) in the real axis.
Notice that $|z|$ is the length of the vector $\vec{\mathrm{OP}}$.

Theorem 7.1: Let z and w be complex numbers. Then

(1) $\bar{\bar{z}} = z$ and $|z| = |\bar{z}|$.

(2) $\bar{z} = z$ if and only if z is real (*i.e.*, is of the form $a + 0i$, for some real number a).

(3) $\overline{z+w} = \bar{z} + \bar{w}$.

(4) $\overline{-z} = -\bar{z}$.

(5) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$.

(6) If $z = a + ib \neq 0$, then $\dfrac{1}{z} = \dfrac{a - ib}{|z|^2}$.

(7) $\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$, if $z \neq 0$.

(8) $|z|^2 = z \cdot \bar{z}$.

(9) $|z \cdot w| = |z| \cdot |w|$.

(10) $|z + w| \leq |z| + |w|$ (Triangle inequality).

(11) $|z| - |w| \leq |z - w|$.

Proof: (1) and (2) are clear. Equations (3) and (5) may be checked by straightforward calculations and (4) and (7) may then be proved by a trick

$$0 = \bar{0} = \overline{z + (-z)} = \bar{z} + \overline{-z}$$

and we can conclude (4)

$$\overline{-z} = -\bar{z}.$$

In the same way

$$1 = \bar{1} = \overline{z \cdot \frac{1}{z}} = \bar{z} \cdot \overline{\frac{1}{z}}$$

and we can conclude (7)

$$\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}},$$

if $z \neq 0$.

(8) and (9) may also be proved by a straightforward calculation.

For (6) we compute

$$z \cdot \left(\frac{a - ib}{|z|^2}\right) = (a + ib) \cdot \left(\frac{a - ib}{|z|^2}\right) = \frac{a^2}{|z|^2} + \frac{b^2}{|z|^2} = \frac{a^2 + b^2}{|z|^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1.$$

In general to find the following for $c + id \neq 0$

$$\frac{a + ib}{c + id}$$

we use the trick

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{(a + ib)(c - id)}{(c^2 + d^2)}.$$

It remains to show (10) and (11). Notice that the following hold

(a) $w \cdot \bar{z} = \overline{z \cdot \bar{w}}$ by (1) and (5).

(b) $u + \bar{u} = 2\mathrm{Re}(u)$ for all complex numbers $u = x + iy$, since

$$u + \bar{u} = (x + iy) + (x - iy) = 2x = 2\mathrm{Re}(u).$$

(c) $\mathrm{Re}(u) \leq |u|$ for all complex numbers $u = x + iy$, since $\mathrm{Re}(u) = x \leq \sqrt{x^2} \leq \sqrt{x^2 + y^2} = |u|$.

(11) follows from (10). We put $z_1 = z - w$ and $z_2 = w$. Then (10) yields

$$|z| = |z_1 + z_2| \leq |z_1| + |z_2| = |z - w| + |w|.$$

Therefore $|z| - |w| \leq |z - w|$, showing (11).
We have

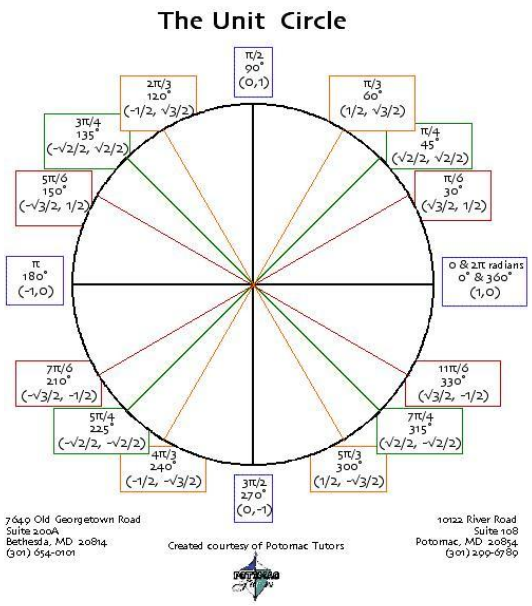
$$\begin{aligned}
|z + w|^2 &\overset{(1)}{=} (z + w)(\overline{z + w}) \overset{(3)}{=} (z + w)(\bar{z} + \bar{w}) \\
&= z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w \\
&= |z|^2 + |w|^2 + z\bar{w} + \overline{z\bar{w}} \quad \text{by} \quad \text{(a)} \\
&= |z|^2 + |w|^2 + 2\mathrm{Re}(z\bar{w}) \quad \text{by} \quad \text{(b)} \\
&\leq |z|^2 + |w|^2 + 2|z\bar{w}| \quad \text{by} \quad \text{(c)} \\
&= |z|^2 + |w|^2 + 2|z| \cdot |w| \quad \text{by} \quad \text{(9) and (1)} \\
&= (|z| + |w|)^2 \, .\square
\end{aligned}$$

Geometrically the inequality (10) merely asserts that the length of side AC of the triangle ABC cannot exceed the sum of the lengths of the sides AB and BC; equality is possible only when A, B and C lie on the same line. Hence the name triangle inequality.

Geometrical representation of the multiplication The geometrical interpretation of multiplication is more involved. We first look at complex numbers $z = a + ib$ with $|z| = \sqrt{a^2 + b^2} = 1$. In the complex plane these are the complex numbers located on the unit circle. In this case, z can be written in the form

$$z = (\cos\theta, \sin\theta) = \cos\theta + i\sin\theta,$$

for some number θ (angle of θ radians).



An arbitrary complex number $w \neq 0$ can be written in the form $w = r \cdot z$ with r a positive real number and z a complex number lying on the unit circle.

Since $w = |w|\frac{w}{|w|}$, this follows with $r = |w|$ and $z = \frac{w}{|w|}$.

To see this notice that if $w = x + iy$, then $\left|\frac{w}{|w|}\right| = \frac{|w|}{|w|} = 1$.
As a result, any non-zero complex number can be written

$$w = r(\cos\theta, \sin\theta) = r(\cos\theta + i\sin\theta),$$

for some real $r > 0$ and for some real number θ (angle of θ radians).
This is called the polar form of the complex number w.

The number r is unique (it equals $|w|$), but θ is not unique; if θ_0 is one possibility, then the others are $\theta_0 + 2\pi n$, for $n \in \mathbb{Z}$.

Definition: Any of the real numbers θ such that $w = r(\cos\theta + i\sin\theta)$, with $r = |w|$ is called an argument of w.

To find an argument θ for $w = x + iy$, we may note that the equation

$$x + iy = w = |w|(\cos\theta, \sin\theta) = |w|(\cos\theta + i\sin\theta)$$

means that $x = |w|\cos\theta$ and $y = |w|\sin\theta$. So, for example, if $x > 0$ we can take

$$\theta = \arctan(\tan\theta) = \arctan\left(\frac{\sin\theta}{\cos\theta}\right) = \arctan\frac{y}{x}.$$

If $x = 0$, we can take $\theta = \frac{\pi}{2}$ when $y > 0$ and $\theta = \frac{3\pi}{2}$ when $y < 0$.

To describe the product of complex numbers geometrically we need an important formula.

Theorem 7.2: For all real numbers u, v:

(a) $\sin(u + v) = \sin u \cdot \cos v + \cos u \cdot \sin v$,

(b) $\cos(u + v) = \cos u \cdot \cos v - \sin u \cdot \sin v$.

Proof: It will be given at the end of this Chapter.

The product of two non-zero complex numbers

$$z = r(\cos\theta + i\sin\theta),$$

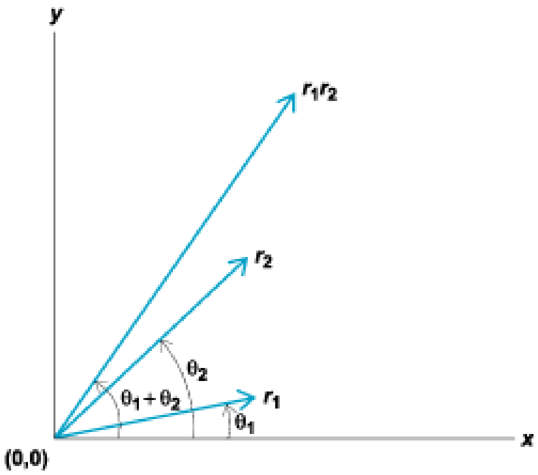
$$w = s(\cos\delta + i\sin\delta),$$

is given by

$$\begin{aligned} z \cdot w &= rs(\cos\theta + i\sin\theta)(\cos\delta + i\sin\delta) \\ &= rs\left[(\cos\theta\cos\delta - \sin\theta\sin\delta) + i(\sin\delta\cos\theta + \sin\theta\cos\delta)\right] \\ &\overset{(a)(b)}{=} rs[\cos(\theta + \delta) + i\sin(\theta + \delta)]. \end{aligned}$$

Thus, the absolute value of a product is the product of the absolute values of the factors, while the sum of any argument for each of the factors will be an argument for the product.

Geometrically, multiplication of two complex numbers z and w with $|z| = |w| = 1$ and arguments θ and δ means that we add the angles θ and δ.



There is an important formula for the powers of complex numbers.

7.3 Powers of complex numbers

De Moivre's theorem 7.3 For all integers $n \geq 0$,

$$z^n = |z|^n(\cos n\theta + i\sin n\theta),$$

for any argument θ of z.

Proof: by Mathematical Induction.
For $n = 0$ the proof is clear.
Now suppose this is true for n. We have to show that it holds for $n + 1$.

$$\begin{aligned} z^{n+1} = z^n \cdot z &= |z|^n(\cos n\theta + i \sin n\theta)|z|(\cos\theta + i\sin\theta) \\ &= |z|^{n+1}[\cos(n\theta + \theta) + i\sin(n\theta + \theta)] \end{aligned}$$

by the computation immediately following Theorem 7.2. We conclude by writing

$$z^{n+1} = |z|^{n+1}[\cos(n + 1)\theta + i \sin(n + 1)\theta].$$

De Moivre's Theorem also holds for negative integers $n < 0$. This can be seen as follows:
Let $k = -n$. Then

$$\begin{aligned} z^n = \frac{1}{z^k} &= \frac{1}{|z|^k(\cos k\theta + i\sin k\theta)} \\ = \frac{(\cos n\theta + i\sin n\theta)}{|z|^k(\cos k\theta + i\sin k\theta)(\cos n\theta + i\sin n\theta)} &= \frac{(\cos n\theta + i\sin n\theta)}{|z|^k(\cos(k\theta + n\theta) + i\sin(k\theta + n\theta)} \\ = \frac{|z|^n(\cos n\theta + i\sin n\theta)}{\cos(k\theta + n\theta) + i\sin(k\theta + n\theta)} &= \frac{|z|^n(\cos n\theta + i\sin n\theta)}{\cos(0) + i\sin(0)} \\ &= |z|^n(\cos n\theta + i\sin n\theta). \end{aligned}$$

Definition: We denote by $\arg(z)$ an argument of the complex number z. Note that $\arg(z)$ is only determined up to integer multiples of 2π. It is necessary to remove this ambiguity, so by convention the value of the argument θ of z is chosen so that it lies in the interval $-\pi < \theta \leq \pi$. This value of θ is called the principal value of the argument and denoted by $\mathrm{Arg}(z)$.

Corollary 7.4: (i) $\arg(z \cdot w) = \arg(z) + \arg(w)$.

(ii) $\arg(z_1 \cdot z_2 \cdot \cdots \cdot z_n) = \arg(z_1) + \cdots + \arg(z_n)$.

(iii) Let $w \neq 0$ $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$.

Proof: (i) Let $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\delta + i\sin\delta)$, where $r, s \in \mathbb{R}$ and $r, s \geq 0$. Then $\arg(z) = \theta$ and $\arg(w) = \delta$. We computed earlier that

$$z \cdot w = rs[\cos(\theta + \delta) + i\sin(\theta + \delta)].$$

We can conclude that $\arg(z \cdot w) = \theta + \delta = \arg(z) + \arg(w)$.

(ii) follows by repeated applications of (i).

(iii) By de Moivre's theorem we have

$$\arg\left(\frac{1}{w}\right) = \arg\left(w^{-1}\right) = -\delta.$$

So by (i) we get

$$\arg\left(\frac{z}{w}\right) = \arg\left(z \cdot \frac{1}{w}\right) = \arg(z) + \arg\left(\frac{1}{w}\right) = \theta - \delta = \arg(z) - \arg(w).$$

Let $z = x + iy$, we want to determine $\text{Arg}(z)$, the principal value of the argument of z.

If $x = 0$ and $y > 0$, then $\text{Arg}(z) = \frac{\pi}{2}$.

If $x = 0$ and $y < 0$, then $\text{Arg}(z) = -\frac{\pi}{2}$.

If $x \geq 0$ and $y = 0$, then $\text{Arg}(z) = 0$.

If $x < 0$ and $y = 0$, then $\text{Arg}(z) = \pi$.

If

1. $x > 0, y > 0$, then $\text{Arg}(z) = \arctan \frac{y}{x}$;
2. $x < 0, y > 0$, then $\text{Arg}(z) = \pi - \arctan \left|\frac{y}{x}\right|$;
3. $x < 0, y < 0$, then $\text{Arg}(z) = \arctan \frac{y}{x} - \pi$;
4. $x > 0, y < 0$, then $\text{Arg}(z) = -\arctan \left|\frac{y}{x}\right|$.

7.3.1 Examples

(i) Find the Cartesian form of the complex number z for which $|z| = 3$ and $\arg(z) = \frac{\pi}{6}$.

Solution: Here $r = 3$ and $\theta = \arg(z) = \frac{\pi}{6}$, so

$$x = 3\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2} \quad \text{and} \quad y = 3\sin\frac{\pi}{6} = \frac{3}{2}.$$

(ii) Find the modulus-argument form of $z = -5 - i5\sqrt{3}$.

Solution: As $z = -5 - i5\sqrt{3}$ we have $x = -5$ and $y = -5\sqrt{3}$, so

$$r = \sqrt{(-5)^2 + (-5\sqrt{3})^2} = \sqrt{100} = 10$$

and since z lies in the third quadrant [*i.e.*, (3) applies] we get

$$\text{Arg}(z) = \arctan\sqrt{3} - \pi = -\frac{2\pi}{3}$$

(iii) Given $z = 2\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$ and $w = 3\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$, find (a) $z \cdot w$ and (b) $\frac{z}{w}$.

Solution:

$$z \cdot w = 2\cdot 3\left[\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\right] = 6\left[\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right],$$

$$\frac{z}{w} = \frac{2}{3}\left[\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\right] = \frac{2}{3}\left[\cos\frac{-\pi}{12} + i\sin\frac{-\pi}{12}\right] = \frac{2}{3}\left[\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right].$$

(iv) Find $(1+i)^{25}$.

Solution: Setting $z = 1 + i$ we see that $r = |z| = \sqrt{2}$ and from rule (1) above for determining Arg(z), we find

$$\theta = \text{Arg}(z) = \arctan 1 = \frac{\pi}{4}.$$

Thus

$$\begin{aligned} z^{25} &= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{25} \\ &\overset{\text{de Moivre}}{=} (\sqrt{2})^{25}\left(\cos\frac{25\pi}{4} + i\sin\frac{25\pi}{4}\right) \\ &= (\sqrt{2})^{25}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = (\sqrt{2})^{25}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 2^{\frac{25}{2}}\cdot 2^{-1}(1+i) = 2^{12}(1+i)\,. \end{aligned}$$

(v) Find $(2\sqrt{3} - 2i)^{30}$.

Solution: Setting $z = 2\sqrt{3} - 2i$ we see that $r = 4$ and from rule (4) above

$$\text{Arg}(z) = -\arctan\left|\frac{-2}{2\sqrt{3}}\right| = -\arctan\frac{1}{\sqrt{3}} = -\frac{\pi}{6}.$$

Thus

$$\begin{aligned} z^{30} &= \left[4\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right)\right]^{30} \\ &\overset{\text{de Moivre}}{=} (4)^{30}\left[\cos\left(\frac{-30\pi}{6}\right) + i\sin\left(\frac{-30\pi}{6}\right)\right] \\ &= (4)^{30}\left[\cos(-5\pi) + i\sin(-5\pi)\right] = (4)^{30}\left[\cos(5\pi) - i\sin(5\pi)\right] \\ &= (4)^{30}\left[\cos(\pi) - i\sin(\pi)\right] = (4)^{30}\cos(\pi) = -4^{30}\,. \end{aligned}$$

(vi) Find $\sin 5\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Solution:
Let $a = \cos\theta$ and $b = \sin\theta$. We have:

$$\cos 5\theta + i\sin 5\theta \overset{\text{de Moivre}}{=} (\cos\theta + i\sin\theta)^5 = (a+ib)^5 \ .$$

We can write

$$\cos 5\theta + i\sin 5\theta = a^5(ib)^0 + 5a^4(ib)^1 + \frac{5\cdot 4}{1\cdot 2}a^3(ib)^2 + \frac{5\cdot 4}{1\cdot 2}a^2(ib)^3 + 5a^1(ib)^4 + a^0(ib)^5.$$

We can write

$$\cos 5\theta + i\sin 5\theta = a^5 + i5a^4b^1 - 10a^3b^2 - i10a^2b^3 + 5a^1b^4 + ib^5.$$

Notice that the previous equation yields

$$\cos 5\theta = a^5 - 10a^3b^2 + 5a^1b^4 \ ,$$

and

$$i\sin 5\theta = +i5a^4b - i10a^2b^3 + ib^5 \ .$$

Equating imaginary parts gives

$$\sin 5\theta = 5a^4b - 10a^2b^3 + b^5 \ .$$

Finally we have

$$\sin 5\theta = 5(\cos\theta)^4\sin\theta - 10(\cos\theta)^2(\sin\theta)^3 + (\sin\theta)^5 \ .$$

7.4 Roots of complex numbers and polynomials

Theorem 7.5: Every non-zero complex number has exactly n complex n^{th} roots. More precisely, for any complex number $w \neq 0$ and any natural number n, there are precisely n different complex numbers z such that $z^n = w$.

Proof: Let $w = s(\cos\theta + i\sin\theta)$ with $s = |w|$ and some number θ. Then a complex number

$$z = r(\cos\delta + i\sin\delta)$$

with $r > 0$ satisfies $z^n = w$ if and only if

$$r^n(\cos n\delta + i\sin n\delta) = s(\cos\theta + i\sin\theta),$$

which happens if and only if

$$r^n = s$$

and

$$(\cos n\delta + i\sin n\delta) = (\cos\theta + i\sin\theta).$$

From the first equation it follows that

$$r = s^{\frac{1}{n}}.$$

From the second equation it follows that for some integer k we have

$$\theta = n\delta - 2k\pi,$$

so that

$$\delta = \delta_k = \frac{\theta}{n} + \frac{2k\pi}{n}.$$

Conversely, if we choose

$$r = s^{\frac{1}{n}}$$

and some δ_k as above, then the number $z = r(\cos\delta_k + i\sin\delta_k)$ will satisfy

$$z^n = w.$$

To determine the number of n^{th} roots of w, it is therefore only necessary to determine which such z are distinct.
Now, any integer k can be written as $k = nq + k^*$ for some integer q and some integer k^* between 0 and $n-1$. Then,

$$(\cos\delta_k + i\sin\delta_k) = (\cos\delta_{k^*} + i\sin\delta_{k^*}),$$

since

$$\cos\delta_k = \cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) = \cos\left(\frac{\theta}{n} + \frac{2nq\pi + 2k^*\pi}{n}\right)$$

$$= \cos\left(\frac{\theta}{n} + 2q\pi + \frac{2k^*\pi}{n}\right) = \cos\left(\frac{\theta}{n} + \frac{2k^*\pi}{n}\right) = \cos\delta_{k^*},$$

and similarly $\sin\delta_k = \sin\delta_{k^*}$. This shows that every z satisfying $z^n = w$ can be written as

$$z = s^{\frac{1}{n}}(\cos\delta_k + i\sin\delta_k) \qquad \text{for} \quad k = 0, 1, 2, \cdots, n-1.$$

However, it is easy to see that these numbers are all different, since any two δ_k for $k = 0, 1, 2, \cdots, n-1$ differ by less than 2π.

Corollary 7.6: Let $w \neq 0$ and n be a natural number. The n distinct roots $z_0, \cdots, z_{n-1}$ of $w = s(\cos\theta + i\sin\theta)$ are given by

$$z_k = \sqrt[n]{s}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right],$$

with $k = 0, 1, \cdots, n-1$. Here $\sqrt[n]{s}$ is the nth positive real root of s.

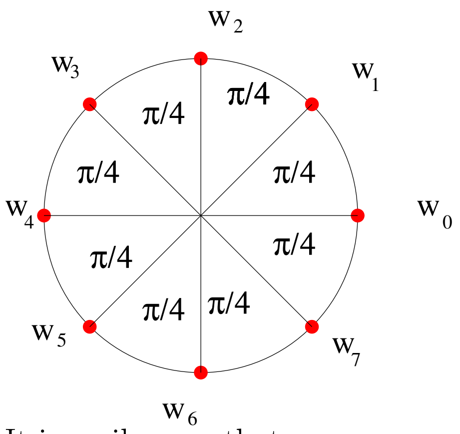
Proof: This was proved in the proof of Theorem 7.5.

7.4.1 Examples:

(1) Find the 8th roots of $w = 1$.
Solution:
$1 = \cos 0 + i \sin 0$, so setting $w = 1$ we see that $r = |w| = 1$ and $\theta = \arg(w) = 0$. Thus the eight roots are

$$z_k = \cos\left(\frac{2k\pi}{8}\right) + i\sin\left(\frac{2k\pi}{8}\right) = \cos\left(\frac{k\pi}{4}\right) + i\sin\left(\frac{k\pi}{4}\right),$$

with $k = 0, 1, \cdots, 7$.
The locations of these points around the unit circle are as follows:



It is easily seen that

$$w_0 = 1 \qquad w_1 = \frac{(1+i)}{\sqrt{2}} \qquad w_2 = i \qquad w_3 = \frac{(-1+i)}{\sqrt{2}}$$

$$w_4 = -1 \qquad w_5 = \frac{(-1-i)}{\sqrt{2}} \qquad w_6 = -i \qquad w_7 = \frac{(1-i)}{\sqrt{2}}.$$

(2) Find the 5th roots of $w = \sqrt{3} - i$.
Solution:
We have $w = 2[\cos(-\pi/6) + i\sin(-\pi/6)]$ and we see that $|w| = 2$ and $\arg(w) = -\pi/6$. Thus, the five roots are seen to be given by

$$z_k = 2^{\frac{1}{5}}\left[\cos\left(\frac{2k\pi - \frac{\pi}{6}}{5}\right) + i\sin\left(\frac{2k\pi - \frac{\pi}{6}}{5}\right)\right],$$

with $k = 0, 1, 2, 3, 4$. Consequently the required roots are

$$z_k = 2^{\frac{1}{5}}\left[\cos\left(\frac{(12k-1)\pi}{30}\right) + i\sin\left(\frac{(12k-1)\pi}{30}\right)\right],$$

with $k = 0, 1, 2, 3, 4$.

(3) Find $i^{\frac{2}{3}}$.

Solution:
If we find the three cube roots of i, that is, if we solve the equation $z^3 = i$, for the numbers z_0, z_1, z_2, the three values of $i^{\frac{2}{3}}$ will be z_0^2, z_1^2, z_2^2.
This follows by setting $z = i^{\frac{1}{3}}$, because then $z^3 = i$, so that $z^2 = i^{\frac{2}{3}}$.
Setting $w = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, it follows that $r = |w| = 1$ and $\theta = \arg(w) = \frac{\pi}{2}$, and so the three cube roots of i are

$$z_k = \cos\left(\frac{2k\pi + \frac{\pi}{2}}{3}\right) + i\sin\left(\frac{2k\pi + \frac{\pi}{2}}{3}\right),$$

with $k = 0, 1, 2$. Hence we can write

$$\begin{aligned} z_0 &= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{1}{2}(\sqrt{3} + i), \\ z_1 &= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = \frac{1}{2}(-\sqrt{3} + i), \\ z_2 &= \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6} = -i, \end{aligned}$$

so the three roots of $i^{\frac{2}{3}}$ are

$$\begin{aligned} z_0^2 &= \frac{1}{2}(1 + i\sqrt{3}), \\ z_1^2 &= \frac{1}{2}(1 - i\sqrt{3}), \\ z_2^2 &= -1. \end{aligned}$$

(4) Let $z \neq 1$ be any one of the n roots of $z^n = 1$. Prove that for any positive integer $n > 1$

$$1 + z + z^2 + \cdots + z^{n-1} = 0.$$

Solution:
Set $s = 1 + z + \cdots + z^{n-1}$ and multiply by z to obtain $zs = z + z^2 + \cdots + z^n$. Thus,

$$zs = z + z^2 + \cdots + z^n \Rightarrow zs = z + z^2 + \cdots + z^n + 1 - 1 \Rightarrow zs - s = z^n - 1$$

$$\Rightarrow s(1 - z) = 1 - z^n \Rightarrow s = \frac{1 - z^n}{1 - z}.$$

However, $z^n = 1$ and $1 - z \neq 0$, we see that $s = 0$ and the result follows.

7.4.2 Polynomials and their root

A fundamental property of $\mathbb{C}$ is that any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ (where the coefficients a_n are complex and $a_n \neq 0$) has a root.

Let c be a root of $p(z)$. Then

$$0 = a_0 + a_1 c + \cdots + a_n c^n,$$

and hence

$$\bar{0} = \bar{a}_0 + \bar{a}_1\bar{c} + \cdots + \bar{a}_n\bar{c}^n = 0.$$

Suppose all a_k are reals. Then

$$a_k = \bar{a}_k,$$

i.e.,

$$0 = a_0 + a_1\bar{c} + \cdots + a_n\bar{c}^n = 0,$$

so $\bar{c}$ is also a root of $p(z)$.

Theorem 7.7: If c is a root of a polynomial with real coefficients, then $\bar{c}$ is also a root.
Proof: We just did it.

7.4.3 Examples

(1) Since i is a root of $z^2 + 1$, the other root will be $\bar{i} = -i$.

(2) Given that $2 - 3i$ is a root of

$$p(z) = z^4 - 6z^3 + 26z^2 - 46z + 65$$

find the other 3 roots.
Since $p(z)$ has real coefficients, another root is $2 + 3i$. We can write

$$p(z) = [z - (2 - 3i)] \cdot [z - (2 + 3i)] \cdot q(z) = (z^2 - 4z + 13) \cdot q(z),$$

for a quadratic polynomial $q(z)$.
By long division we have

$$\begin{array}{llllll|l}
z^4 & -6z^3 & +26z^2 & -46z & +65 & & z^2 - 4z + 13 \\ \hline
-z^4 & +4z^3 & -13z^2 & & & & z^2 - 2z + 5 \\
 & -2z^3 & +13z^2 & -46z & +65 & & -2z \\
 & 2z^3 & -8z^2 & +26z & & & \\
 & & 5z^2 & -20z & +65 & & \\
 & & -5z^2 & +20z & -65 & . &
\end{array}$$

Thus $q(z) = z^2 - 2z + 5$. Solving $q(z) = 0$ gives

$$z = 1 \pm \sqrt{1-5} = 1 \pm \sqrt{-4} = 1 \pm 2\sqrt{-1} = 1 \pm 2i.$$

So the four roots are $2 \pm 3i$ and $1 \pm 2i$.

Recall that every quadratic equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

can be solved to give

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Of course, if $b^2 - 4ac < 0$ there are no real solutions. But you should keep in mind that there are always solutions in $\mathbb{C}$.

7.5 Exponential function, trigonometric functions and the hyperbolic functions

The exponential function exp is actually defined by a so-called power series via

$$e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

More precisely, e^x is the limit of the sequence $\{s_n\}$, where

$$s_n = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

Similarly, the functions cos and sin can be defined by a power series:

$$\begin{aligned}
\cos x &= \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots . \\
\sin x &= \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .
\end{aligned}$$

These functions can be defined on the complex numbers as well. For complex z we define

$$\begin{aligned}
e^z &= \sum_{k=0}^{+\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\
\cos z &= \sum_{k=0}^{+\infty} (-1)^k \frac{z^{2k}}{(2k)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \\
\sin z &= \sum_{k=0}^{+\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots
\end{aligned}$$

If we replace z by iz in the series for e^z and make a rearrangement of the terms something particularly interesting happens:

$$\begin{aligned}
e^{iz} &= 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \cdots \\
&= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} + \cdots \\
&= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) \\
&= \cos z + i \sin z.
\end{aligned}$$

It is clear from the definition (*i.e.*, the power series) that $\sin(-z) = -\sin z$ and that $\cos(-z) = \cos z$. So we also have

$$e^{-iz} = e^{i(-z)} = \cos z - i \sin z.$$

For complex numbers z and w we also have the functional equation

$$e^z \cdot e^w = e^{z+w}.$$

If we write a complex number in Cartesian form, $z = x + iy$, we thus arrive at

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y),$$

so that $|e^z| = e^x$ and $\arg(e^z) = y$.
From the equations for e^{iz} and e^{-iz} we can derive the formulas (Euler's equations)

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{7.1}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{7.2}$$

The development of complex power series thus places the exponential function at the very core of the development of the elementary functions – it reveals a connection between the trigonometric and exponential functions which was never imagined when these functions were first defined and which could have never been discovered without the use of complex numbers.

Recall that we defined

$$\begin{aligned}\sinh x &= \frac{e^x - e^{-x}}{2}\\ \cosh x &= \frac{e^x + e^{-x}}{2}\end{aligned}$$

for real numbers x.

These functions can be extended to the complex numbers by letting

$$\begin{aligned}\sinh z &= \frac{e^z - e^{-z}}{2}\\ \cosh z &= \frac{e^z + e^{-z}}{2}\end{aligned}$$

From the complex view point the trigonometric and the hyperbolic functions are closely related

$$\begin{aligned}\sinh(iz) &= \frac{e^{iz} - e^{-iz}}{2} \overset{(7.1)}{=} i \sin z\\ \cosh(iz) &= \frac{e^{iz} + e^{-iz}}{2} \overset{(7.2)}{=} \cos z\\ \sin(iz) &\overset{(7.1)}{=} \frac{e^{-z} - e^{z}}{2i} = (-i)\frac{e^{-z} - e^{z}}{2} = (i)\frac{e^{z} - e^{-z}}{2} = i \sinh z\\ \cos(iz) &\overset{(7.2)}{=} \frac{e^{-z} + e^{z}}{2} = \cosh z\end{aligned}$$

Thus, we have just shown that

$$\begin{aligned}
\sin z &= \frac{\sinh(iz)}{i} \\
\cos z &= \cosh(iz) \\
\sinh z &= \frac{\sin(iz)}{i} \\
\cosh z &= \cos(iz)
\end{aligned}$$

Theorem 7.8: For all complex numbers z, w:

(a) $\sin(z+w) = \sin z \cos w + \cos z \sin w$,

(b) $\cos(z+w) = \cos z \cos w - \sin z \sin w$.
Proof: Use equations (7.1) and (7.2).

Remark: Certain properties of exp and the trigonometric functions that hold for real arguments are no longer true of complex arguments. For example, exp takes on every complex value except 0 and sin takes on every complex value. In particular, there are complex numbers z and w such that $e^z = -50000$ and $\sin z = 50000$.

Chapter 8

Partial differentiation

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8.1 Many-variable functions

In the previous chapters we have restricted ourselves to the study of functions of only one variable x: $f : A \longrightarrow B$, with **domain** $A \subset \mathbb{R}$ and **range** $B \subset \mathbb{R}$, *i.e.,* for all $x \in A$, $f(x) \in B$.

Let us suppose we need to build the function that associates to each point in space its temperature.

A point in space is determined by three variables $(x, y, z) \in \mathbb{R}^3$. We define the function temperature f that associates to each point in space its temperature $f(x, y, z)$ as follows:

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R} \quad f(x, y, z) = x \cdot y \cdot z \in \mathbb{R}.$$

Note that the function f depends on three variables x, y, z. f is a real-valued function of three real variables.

Definition (provisional): A many-variable (n) function is a rule which assigns, to n real numbers in its domain, some real number in its range. An n-variable function f is defined as follows:

$$f : \mathbb{R}^n \longrightarrow \mathbb{R} \quad f(x_1, x_2, \cdots, x_n) \in \mathbb{R}.$$

8.1.1 Examples of many-variable functions

1. $f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f_1(x,y) = x^2y + xy^2$.

2. $f_2 : [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$, $f_2(x,y) = \sqrt{xy}$.

3. $f_3 : (0,+\infty) \times (0,+\infty) \times (0,+\infty) \longrightarrow (0,+\infty)$, $f_3(x,y,z) = \frac{1}{xyz}$.

4.
$$f_4 : \mathbb{R}^3 \longrightarrow \mathbb{R}, f_4(x,y,z) = \begin{cases} 0 & if & x & \text{is rational} & y & \text{is rational} & z & \text{is rational} \\ 1 & if & x & \text{is rational} & y & \text{is rational} & z & \text{is irrational} \\ 2 & if & x & \text{is rational} & y & \text{is irrational} & z & \text{is rational} \\ 3 & if & x & \text{is rational} & y & \text{is irrational} & z & \text{is irrational} \\ 4 & if & x & \text{is irrational} & y & \text{is rational} & z & \text{is rational} \\ 5 & if & x & \text{is irrational} & y & \text{is rational} & z & \text{is irrational} \\ 6 & if & x & \text{is irrational} & y & \text{is irrational} & z & \text{is rational} \\ 7 & if & x & \text{is irrational} & y & \text{is irrational} & z & \text{is irrational} \end{cases}$$

5.
$$f_5 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}, f_5(x,y) = \tan(x) \cdot \tan(y) = \frac{\sin(x)}{\cos(x)} \frac{\sin(y)}{\cos(y)}.$$

8.1.2 Building new functions

Suppose
$$f, g : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}.$$
Define
$$f + g : A \longrightarrow \mathbb{R}$$
by $(f+g)(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n) + g(x_1, x_2, \cdots, x_n)$, the sum of f and g.

Define
$$f - g : A \longrightarrow \mathbb{R}$$
by $(f-g)(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n) - g(x_1, x_2, \cdots, x_n)$, the difference of f and g.

Define
$$(f \cdot g) : A \longrightarrow \mathbb{R}$$
by $(f \cdot g)(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n) \cdot g(x_1, x_2, \cdots, x_n)$, the product of f and g.

If $g(x_1, x_2, \cdots, x_n) \neq 0$ for all $(x_1, x_2, \cdots, x_n) \in A$, we also define
$$\left(\frac{f}{g}\right) : A \longrightarrow \mathbb{R}$$
by $\left(\frac{f}{g}\right)(x_1, x_2, \cdots, x_n) = \frac{f(x_1, x_2, \cdots, x_n)}{g(x_1, x_2, \cdots, x_n)}$, the quotient of f and g.

8.2 Limits and continuity

Provisional definition: The function f will be said to have the **limit** L as $x = (x_1, x_2, \cdots, x_n)$ tends to $a = (a_1, a_2, \cdots, a_n)$, if when $x \in \mathbb{R}^n$ is arbitrarily close to, but unequal to $a \in \mathbb{R}^n$, $f(x)$ is arbirtrarily close to L.

The statement "tends to a" is written as $x \to a$, and when the limit of $f(x)$ exists as $x \to a$, this will be shown by writing

$$\lim_{x\to a} f(x) = L.$$

8.2.1 Elementary properties of limits

Suppose $x \in A$, $a \in \mathbb{R}^n$, that

$$f, g : A \subset \mathbb{R}^n \longrightarrow \mathbb{R},$$

and that

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M.$$

Then we have

(1) $\lim_{x\to a}[b \cdot f(x)] = b \cdot L$, with $b \in \mathbb{R}$.

(2) $\lim_{x\to a}[f(x) \pm g(x)] = L \pm M$.

(3) $\lim_{x\to a}[f(x) \cdot g(x)] = L \cdot M$.

(4) If $M \neq 0$, $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$.

Example: Find

$$\lim_{(x,y)\to(2,2)} \left[\frac{x^2+5x+3}{2x^3-x+4}\right]\left[\frac{y^2+5y+3}{2y^3-y+4}\right].$$

We can write

$$\lim_{(x,y)\to(2,2)} \left[\frac{2^2+5\cdot 2+3}{2\cdot 2^3-2+4}\right]\left[\frac{2^2+5\cdot 2+3}{2\cdot 2^3-2+4}\right] = \frac{17}{18}\cdot\frac{17}{18}.$$

Example: Find

$$\lim_{(x,y)\to(1,1)} \left[\frac{2x^2+x-3}{x^2+x-2}\right]\left[\frac{2y^2+y-3}{y^2+y-2}\right].$$

We can write

$$\lim_{(x,y)\to(1,1)} \left[\frac{2x^2+x-3}{x^2+x-2}\right]\left[\frac{2y^2+y-3}{y^2+y-2}\right] = \lim_{(x,y)\to(1,1)} \left[\frac{(x-1)(2x+3)}{(x-1)(x+2)}\right]\left[\frac{(y-1)(2y+3)}{(y-1)(y+2)}\right]$$

$$= \lim_{(x,y)\to(1,1)} \left[\frac{(2x+3)}{(x+2)}\right]\left[\frac{((2y+3)}{(y+2)}\right] = \frac{5}{3}\cdot\frac{5}{3}.$$

8.2.2 Continuity

If f is an arbitrary function, it is not necessarily true that

$$\lim_{x \to a} f(x) = f(a).$$

Definition: Suppose $x \in A$, $a \in \mathbb{R}^n$ and that

$$f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}.$$

The function f is **continuous at** a if

$$\lim_{x \to a} f(x) = f(a).$$

Without lack of generality we will restrict ourselves to functions of two real variables in most of what follows.

8.3 Differentiation

We recall at this point our previous definition of differentiable function:

Definition: The function f is differentiable at a if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists.
We define, $f'(a)$ to be that limit, namely,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \ .$$

For a function of two variables $f(x, y)$ we may define the derivative with respect to either x or y.

We consider the derivative with respect to x, for example, by saying that it is that for a one-variable function when y is held fixed and treated as a constant (or to y, for example, by saying that it is that for a one-variable function when x is held fixed and treated as a constant).

To signify that a derivative is with respect to x, but at the same time to recognise that a derivative with respect to y also exists, the former is denoted by $\partial f / \partial x$ and is called the partial derivative of $f(x, y)$ with respect to x.

Similarly, the partial derivative of $f(x, y)$ with respect to y is denoted by $\partial f / \partial y$.

Definition: We define the partial derivative of $f(x, y)$ with respect to x as

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h},$$

provided that the limit exists.

We define the partial derivative of $f(x, y)$ with respect to y as

$$\frac{\partial f(x,y)}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h},$$

provided that the limit exists.

The extension to the general $n-$variable case is given below.

We define the partial derivative of $f(x_1, x_2, \cdots, x_n)$ with respect to x_k as

$$\frac{\partial f(x_1,x_2,\cdots,x_n)}{\partial x_k} = \lim_{h\to 0} \frac{f(x_1,x_2,\cdots,x_k+h,\cdots,x_n) - f(x_1,x_2,\cdots,x_k,\cdots,x_n)}{h},$$

provided that the limit exists.

8.3.1 Partial differentiation for two-variable functions

For a two-variable function $f(x, y)$, we can compute two first partial derivatives:

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y},$$

and four first partial derivatives:

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial x^2} = f_{xx} \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial y \partial x} = f_{yx} \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial y^2} = f_{yy}
\end{aligned}$$

Theorem: If the second partial derivatives of $f(x, y)$ are continuous at all points (x, y) in its domain, we have

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Example: Find the first and second partial derivatives of the function

$$f(x,y) = 2x^3y^2 + y^3.$$

Solution:

$$\frac{\partial f(x,y)}{\partial x} = 2\cdot 3x^2 \cdot y^2,$$

$$\frac{\partial f(x,y)}{\partial y} = 2\cdot x^3 \cdot 2\cdot y + 3y^2,$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = 2\cdot 3\cdot 2x\cdot y^2,$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = 2\cdot x^3\cdot 2 + 3\cdot 2y,$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x} = 2\cdot 3x^2\cdot 2y$$

8.4 Stationary values of many-variable functions

Reminder: We recall that a function $f(x)$ of one variable has a stationary point at $x = x_0$ if

$$\left.\frac{df}{dx}\right|_{x=x_0} = 0.$$

A stationary point is:

(i) a minimum if $df^2/dx^2 > 0$ at $x = x_0$.

(ii) a maximum if $df^2/dx^2 < 0$ at $x = x_0$.

(iii) a point of inflection if $df^2/dx^2 = 0$ at $x = x_0$.

We now consider the stationary points of functions of two variables.

Definition: The function $f(x,y)$ has a stationary point (minimum, maximum or saddle point) at (x_0, y_0) if

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x_0,y_0)} = 0 \quad \text{and} \quad \left.\frac{\partial f(x,y)}{\partial y}\right|_{(x_0,y_0)} = 0.$$

We now turn our attention to determining the nature of a stationary point of a function of two variables, *i.e.,* whether it is a maximum, a minimum or a saddle point.

Definition: The function $f(x, y)$ has a stationary point at (x_0, y_0) if $f_x = f_y = 0$ at (x_0, y_0). The stationary points may be classified further as follows:

(i) minima if both f_{xx} and f_{yy} are positive and $f_{xy}^2 < f_{xx}f_{yy}$.

(ii) maxima if both f_{xx} and f_{yy} are negative and $f_{xy}^2 < f_{xx}f_{yy}$.

(iii) saddle points if f_{xx} and f_{yy} have opposite signs or $f_{xy}^2 > f_{xx}f_{yy}$.

Example: Show that the function $f(x, y) = x^3 \cdot \exp(-x^2 - y^2)$ has a maximum at the point $(\sqrt{3/2}, 0)$, a minimum at $(-\sqrt{3/2}, 0)$ and a stationary point at $(0, 0)$ which nature cannot be determined by the above procedures.
Solution: Do at home.

Example: Find and evaluate the maxima, minima and saddle points of the function $f(x, y) = xy(x^2 + y^2 - 1)$.
Solution:
We compute

$$f_x = y(x^2 + y^2 - 1) + xy(2x) \quad f_y = x(x^2 + y^2 - 1) + xy(2y)$$

$$f_{xx} = y(2x) + y(4x) = 6xy \quad f_{yy} = x(2y) + x(4y) = 6xy$$
$$f_{xy} = (x^2 + y^2 - 1) + y(2y) + x(2x) = 3x^2 + 3y^2 - 1.$$

We solve for $f_x = 0$ and $f_y = 0$. The solutions are

$$(0, 0), \qquad (\pm 1, 0), \qquad (0, \pm 1), \qquad \pm(1/2, 1/2), \qquad \pm(1/2, -1/2).$$

We classify them as follows:

$$\begin{aligned}
(0, 0) &\Rightarrow f_{xy}^2 = 1 > f_{xx}f_{yy} = 0 \Rightarrow \text{saddle point},\\
(\pm 1, 0) &\Rightarrow f_{xy}^2 = 4 > f_{xx}f_{yy} = 0 \Rightarrow \text{saddle point},\\
(0, \pm 1) &\Rightarrow f_{xy}^2 = 4 > f_{xx}f_{yy} = 0 \Rightarrow \text{saddle point},\\
\pm(1/2, 1/2) &\Rightarrow f_{xx} = \frac{3}{2} > 0 \quad \text{and} \quad f_{yy} = \frac{3}{2} > 0 \Rightarrow \text{minimum},\\
\pm(1/2, -1/2) &\Rightarrow f_{xx} = -\frac{3}{2} < 0 \quad \text{and} \quad f_{yy} = -\frac{3}{2} < 0 \Rightarrow \text{maximum}.
\end{aligned}$$

Chapter 9

Sequences

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9.1 Introduction

We now start the "mathematical analysis" part of the module. For the moment we shall be concerned with looking at infinite series more closely, and this will require the notion of a sequence and the limit of a sequence.

We have already seen that there are infinite series for e^x, $\cos x$, $\sin x$, etc. and they come up also as solutions of certain differential equations. The most important idea that comes up first with an infinite series is whether it can be said to have a sum, whether it converges. Often we are more interested in having a sum, rather than knowing what the sum is.

Here is a simple example which shows that infinite sums can behave differently than finite sums and why we need to be rather cautious. For the moment just take it as correct that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

The following manipulations lead to quite a paradoxical result:

$$\begin{aligned} \log 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots \\ &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \cdots \end{aligned}$$

the pattern here is one positive followed by two negative ones. We group terms as follows:

$$\begin{aligned}
\log 2 &= \left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)-\frac{1}{16}+\cdots \\
&= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}+\cdots \\
&= \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\cdots\right) \\
&= \frac{1}{2}\log 2,
\end{aligned}$$

so $\log 2 = (\log 2)/2$, implying that $\log 2 = 0$. But that is not true.

This contradiction depends on a step which takes for granted that operations valid for finite sums necessarily have analogues for infinite sums. Note that the numbers of the second infinite sum are a rearrangement of the numbers of the first sum. So what can happen is that if the order of the terms of an infinite sum is changed, we get two different outcomes. In this respect infinite sums can behave very differently from finite sums.

Before we can start over investigations of infinite series we should study infinite sequences first since the "sum" of an infinite series is defined as a sequence of approximations to its "sum".

The idea of an infinite sequence is so natural a concept that it is tempting to dispense with a definition altogether. One frequently writes simply "an infinite sequence"

$$a_1, a_2, a_3, a_4, \ldots,$$

the three dots indicating that the numbers a_i continue to the right "forever". The important point about an infinite sequence is that for each natural number n, there is a real number a_n. This sort of correspondence is precisely what functions are meant to formalise.

9.2 Sequences

Definition 9.1: An infinite sequence of real numbers is a function whose domain is $\mathbb{N}$.
From the point of view of this definition, a sequence should be designated by a single better like a and particular values by

$$a(1), a(2), a(3), \ldots$$

but the subscript notation

$$a_1, a_2, a_3, \ldots$$

is almost always used instead, and the sequence is usually denoted by a symbol like

$$\{a_n\}.$$

Thus $\{n\}$, $\{(-1)^n\}$ and $\{1/n\}$ denote the sequences α, β, γ defined by

$$\alpha_n = n, \quad \beta_n = (-1)^n \quad \text{and} \quad \gamma_n = \frac{1}{n}.$$

Definition 9.2: A sequence $\{a_n\}$ converges to L, in symbols

$$\lim_{n\to+\infty} a_n = L,$$

if for every $\epsilon > 0$ there is natural number N such that, for all natural numbers n, if

$$n > N, \quad \text{then} \quad |a_n - L| < \epsilon.$$

In addition to the terminology introduced in this definition, we sometimes say that the sequence $\{a_n\}$ approaches L or has the limit L.

A sequence $\{a_n\}$ is said to converge if it converges to L for some L, and to diverge if it does not converge.

To show that the sequence $\{\gamma_n\}$ converges to 0 it suffices to observe that for every $\epsilon > 0$ there exists a natural number N such that $\epsilon > \frac{1}{N}$. Then if $n > N$ we have

$$\gamma_n = \frac{1}{n} < \frac{1}{N} < \epsilon, \quad \text{so} \quad |\gamma_n - 0| < \epsilon.$$

Examples 9.3: Let $\{\delta_n = 2 + (-1)^n \cdot \frac{1}{2^n}\}$ and $\{a_n = \sqrt{n+1} - \sqrt{n}\}$. Show that

(i) $\lim_{n\to+\infty} \delta_n = 2$,

(ii) $\lim_{n\to+\infty} a_n = 0$.

Proof:

(i) Let $\epsilon > 0$. Then $\epsilon > \frac{1}{N} > 0$ for some natural number N. Then, if $n > N$ we have $2^n > n$, so that $\epsilon > \frac{1}{2^n}$ and hence

$$|\delta_n - 2| = \left|(-1)^n \cdot \frac{1}{2^n}\right| = \left|\frac{1}{2^n}\right| = \frac{1}{2^n} < \epsilon.$$

(ii) To estimate $\sqrt{n+1} - \sqrt{n}$ we can use an algebraic trick

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Now let $\epsilon > 0$. Pick K such that $\epsilon > \frac{1}{K}$. Let $N = K^2$. Then for $n > N$ we get $\sqrt{n} > \sqrt{N} = K$, so that

$$|a_n - 0| = |a_n| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}} < \frac{1}{K} < \epsilon.$$

Remark: Above we used the fact that for every $\epsilon > 0$ there exists a natural number N such that $\epsilon > \frac{1}{N}$. How do we find N? If ϵ is rational, *i.e.,* $\epsilon = \frac{p}{q}$ for some natural numbers p, q, we can just pick $N = q + 1$. If $0 < \epsilon < 1$ and $\epsilon = 0.a_1 a_2 a_3 \cdot \ldots$ in decimal expansion, we can let $N = (a_k + 1) \cdot 10^k$, where a_k is the first term with $a_k \neq 0$.

What does it mean geometrically that

$$\lim_{n \to +\infty} b_n = L?$$

If $\epsilon > 0$ then we obtain an open interval $(L - \epsilon, L + \epsilon) = I$ with L being its centre.

Since $\lim_{n \to +\infty} b_n = L$, there exists a natural number N such that, for all $n > N$, $|b_n - L| < \epsilon$. This means that, if $n > N$, then b_n is in the open interval I. Thus almost all terms of $\{b_n\}$ lie in the ϵ-neighbourhood of L.

Here almost all means all, except for finitely many.

Except for very simple examples like those from 9.3, it can be difficult to verify the limit of a sequence directly from the definition. So we have to prove some general theorems which bypass this difficulty and make the calculations easier. Note also that not every sequence has a limit, *e.g.,* $\{n\}$.

The usefulness of the limit concept depends partly on the fact that if a limit exists, then it is unique.

Theorem 9.4: If a sequence has a limit, then that limit is unique.
Proof: Suppose

$$\lim_{n \to +\infty} a_n = L \qquad \lim_{n \to +\infty} a_n = M.$$

Aiming at a contradiction, suppose $L \neq M$. Choose $\epsilon = |L - M|/4$, so $\epsilon > 0$. Then there exists N_1 and N_2 such that

$$|a_n - L| < \epsilon \quad \text{for all} \quad n > N_1 \quad \text{and} \quad |a_n - M| < \epsilon \quad \text{for all} \quad n > N_2.$$

Thus, for all $n > max(N_1, N_2)$, we have

$$|L - M| = |L - a_n + a_n - M| \leq |L - a_n| + |a_n - M| \leq 2\epsilon,$$

so that $4\epsilon \leq 2\epsilon$ which yields the absurd result that $4 \leq 2$.

Definition 9.5: A sequence $\{a_n\}$ is bounded if there is a constant C such that $|a_n| < C$ holds for all n.
If a sequence is not bounded it is said to be unbounded.

Examples: The sequence $\{a_n\}$ given by

$$\left\{-1 - \frac{1}{2}, 1 + \frac{1}{3}, -1 - \frac{1}{4}, 1 + \frac{1}{5}, \ldots\right\} = \left\{(-1)^n + (-1)^n \frac{1}{n+1}\right\}$$

is bounded by $C = 2$.
On the other hand, the sequence $\{a_n\} = \{n\}$ is unbounded.

Theorem 9.6: A convergent sequence is bounded.
Proof: Suppose

$$\lim_{n\to+\infty} a_n = L.$$

Then there exists N such that $n > N \Rightarrow |a_n - L| < 1$. Let

$$A = \max(|a_1 - L|, |a_2 - L|, \ldots, |a_n - L|).$$

As a result, $|a_n - L| < A + 1$ holds for all n. Therefore we have

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < A + 1 + |L|.$$

So with $C = A + 1 + |L|$ it follows that

$$|a_n| < C \qquad \text{for all} \qquad n.$$

The converse is false: the sequence

$$\{0, 1, 0, 1, 0, 1, \ldots\}$$

is bounded but not convergent.

Notation: To simplify notation we will often write

$$a_n \to L$$

rather than

$$\lim_{n\to+\infty} a_n = L.$$

To make our life easier we better develop some machinery.

Theorem 9.7: If

$$\lim_{n\to+\infty} a_n \qquad \text{and} \qquad \lim_{n\to+\infty} b_n$$

both exist, then
(i)

$$\lim_{n\to+\infty} (a_n + b_n) = \lim_{n\to+\infty} a_n + \lim_{n\to+\infty} b_n.$$

(ii)

$$\lim_{n\to+\infty} (a_n \cdot b_n) = \lim_{n\to+\infty} a_n \cdot \lim_{n\to+\infty} b_n.$$

(iii) Moreover, if $\lim_{n\to+\infty} b_n \neq 0$, then $b_n \neq 0$ for all $n > N$ for some N, and

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to+\infty} a_n}{\lim_{n\to+\infty} b_n}.$$

Proof: Let

$$\lim_{n\to+\infty} a_n = L \qquad \text{and} \qquad \lim_{n\to+\infty} b_n = M.$$

(i) Let $\epsilon > 0$. Then there exist N_1, N_2 such that $|a_n - L| < \frac{\epsilon}{2}$ for all $n > N_1$ and $|b_n - M| < \frac{\epsilon}{2}$ for all $m > N_2$.
Set $N = \max(N_1, N_2)$. Then, for all $n > N$

$$|(a_n + b_n) - (L + M)| = |(a_n - L) + (b_n - M)| < |(a_n - L)| + |(b_n - M)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence

$$\lim_{n\to+\infty}(a_n + b_n) = L + M.$$

Note that we used the triangle inequality.
The proofs of (ii) and (iii) are harder than that for (i) and are not given.

Corollary 9.8:

(i) If $\lim_{n\to+\infty} a_n$ exists and c is a constant, then $\{ca_n\}$ converges too and

$$\lim_{n\to+\infty} ca_n = c \lim_{n\to+\infty} a_n.$$

(ii) If $\lim_{n\to+\infty} a_n$ and $\lim_{n\to+\infty} b_n$ both exist, then

$$\lim_{n\to+\infty}(a_n - b_n) = \lim_{n\to+\infty} a_n - \lim_{n\to+\infty} b_n.$$

(iii) If K is a natural number $K > 1$, then

$$\lim_{n\to+\infty} \frac{1}{n^K} = 0.$$

Proof:
(i) The sequence $\{b_n\}$ with $b_n = c$ converges to c. Thus, by Theorem 9.7 (ii) we get

$$\lim_{n\to+\infty} ca_n = \lim_{n\to+\infty} b_n \cdot a_n = \lim_{n\to+\infty} a_n \cdot \lim_{n\to+\infty} b_n = c \lim_{n\to+\infty} a_n.$$

(ii) By (i) we have

$$\lim_{n\to+\infty} -b_n = -\lim_{n\to+\infty} b_n.$$

From this we arrive at (ii) using Theorem 9.7 (i).
(iii) We obviously have

$$\lim_{n\to+\infty} \frac{1}{n} = 0.$$

Thus, using Theorem 9.7 we get

$$\lim_{n\to+\infty} \frac{1}{n^2} = \lim_{n\to+\infty} \frac{1}{n} \cdot \lim_{n\to+\infty} \frac{1}{n} = 0 \cdot 0 = 0.$$

Therefore,

$$\lim_{n\to+\infty} \frac{1}{n^3} = \lim_{n\to+\infty} \frac{1}{n} \cdot \lim_{n\to+\infty} \frac{1}{n^2} = 0 \cdot 0 = 0, \qquad \text{etc.}$$

9.2.1 Examples

(1) Let

$$a_n = \frac{3n^3 + 2n^2 + 13n}{2n^3 + 16n^2 + 5}.$$

Then

$$a_n = \frac{3 + \frac{2}{n} + \frac{13}{n^2}}{2 + \frac{16}{n} + \frac{5}{n^3}}.$$

By theorem 9.8 we get

$$\lim_{n\to+\infty} \frac{2}{n} = 2 \lim_{n\to+\infty} \frac{1}{n} = 2 \cdot 0 = 0$$

and likewise

$$\lim_{n\to+\infty} \frac{13}{n^2} = 0, \qquad \frac{16}{n} \to 0 \qquad \text{and} \qquad \frac{5}{n^3} \to 0.$$

Therefore, using Theorem 9.11(i) we get

$$3 + \frac{2}{n} + \frac{13}{n^2} \to 3 \qquad \text{and} \qquad 2 + \frac{16}{n} + \frac{5}{n^3} \to 2.$$

By 9.7 (iii) we thus get

$$\lim_{n\to+\infty} a_n = \frac{\lim_{n\to+\infty}\left(3 + \frac{2}{n} + \frac{13}{n^2}\right)}{\lim_{n\to+\infty}\left(2 + \frac{16}{n} + \frac{5}{n^3}\right)} = \frac{3}{2}.$$

(2) Let

$$b_n = \frac{n^2 - 2}{n^8 + 6n^3 - 3n + 8}.$$

Then

$$b_n = \frac{\frac{1}{n^6} - \frac{2}{n^8}}{1 + \frac{6}{n^5} - \frac{3}{n^7} + \frac{8}{n^8}} \to \frac{0}{1} = 0.$$

(3) The sequence

$$\frac{1}{\sqrt{n}} \to 0$$

since

$$\frac{1}{\sqrt{n}} < \epsilon$$

whenever

$$n > \frac{1}{\epsilon^2}.$$

(4) Does

$$\lim_{n\to+\infty}(\sqrt{n^2+1} - n)$$

exist?

$$(\sqrt{n^2+1} - n) = \frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{(\sqrt{n^2+1} + n)} = \frac{n^2 + 1 - n^2}{(\sqrt{n^2+1} + n)} = \frac{1}{(\sqrt{n^2+1} + n)} < \frac{1}{n}.$$

Thus the limit is 0.

9.3 Divergence and convergence of sequences

If a sequence is not convergent then it is divergent. There are 4 types of divergence but we shall just concentrate on 2 main ones which are closely related.

Note first that if $\{a_n\}$ is unbounded, then it is divergent (as we have shown that convergent $\Rightarrow$ bounded by Theorem 9.6).

The converse is false: the sequence

$$\{0, 1, 0, 1, 0, 1, \ldots\}$$

is divergent but not not unbounded.

In our first main type of divergence, the terms just keep on getting larger and larger and we can make them as large as we please.

Definition 9.11: If for all $K > 0$ there is an integer N such that $n > N \Rightarrow a_n > K$, we say that $\{a_n\}$ diverges to $+\infty$ (in symbols)

$$a_n \to +\infty.$$

Examples 9.12:

(1) $\{n\}, \{\sqrt{n}\}$, in fact $\{n^r\}$ for any $r > 0$. For example, $\{n^{\frac{1}{4}}\}$ is divergent to $+\infty$ since given any $K > 0$, $n^{\frac{1}{4}} > K$, when $n > K^4$.

(2)

$$a_n = \frac{n^4 - 3n^2 + 2}{6n + 5} = \frac{n^3 - 3n + \frac{2}{n}}{6 + \frac{5}{n}} = \frac{n(n^2 - 3) + \frac{2}{n}}{6 + \frac{5}{n}} \to +\infty.$$

(3) Define

$$b_n = \begin{cases} 2 & \text{if} \quad n \quad \text{is odd} \\ n & \text{if} \quad n \quad \text{is even} \end{cases}$$

so that $b_1 = 2, b_2 = 2, b_3 = 2, b_4 = 4, b_5 = 2, b_6 = 6, \ldots$.

$\{b_n\}$ diverges, but not to $+\infty$ because of the ever present 2.

(4) Let $c_n = (-1)^n n$. So $c_1 = -1, c_2 = 2, c_3 = -3, \ldots$.

$\{c_n\}$ is divergent, but not to $+\infty$ since there is no N such that

$$c_n > 10 \quad \text{for all} \quad N.$$

There is a similar notion of divergence to $-\infty$.

Definition 9.13: $\{a_n\}$ diverges to $-\infty$ (in symbols)

$$a_n \to -\infty.$$

if given $M < 0$, there exists a natural number N such that

$$n > N \Rightarrow a_n < M.$$

Note that $a_n \to -\infty \iff -a_n \to +\infty$.

Examples 9.14: $\{-n\}$, $\{-3n^2 + 15n\}$, etc.

Definition 9.15: A sequence $\{a_n\}$ is increasing (decreasing) if $a_{n+1} \geq a_n$ ($a_{n+1} \leq a_n$) for all n.
A sequence $\{a_n\}$ is strictly increasing (strictly decreasing) if $a_{n+1} > a_n$ ($a_{n+1} < a_n$) for all n.
Examples 9.16:

(1) $1, 1, 2, 2, 3, 3, \ldots$ increasing.

(2) $1, 0, 0, 0, \ldots$ decreasing.

(3) $1, 2, 3, 4, \ldots$ strictly increasing.

(4) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ strictly decreasing.

Theorem 9.17: If a sequence $\{a_n\}$ is bounded and it is increasing (decreasing), then it has a limit.
Proof: This is a result about the construction of real number that cannot be proved in this course.
Examples 9.18:

(1) Set $a_n = 1 + \dfrac{n}{n+1}$. Thus, $a_1 = 1 + 1/2$,$a_2 = 1 + 2/3$, $a_3 = 1 + 3/4$. We conclude that $\{a_n\}$ is strictly increasing.

$$a_n = 1 + \frac{n}{n+1} = 1 + \frac{n(n+2)}{(n+1)(n+2)} = 1 + \frac{n^2 + 2n}{(n+1)(n+2)},$$

$$a_{n+1} = 1 + \frac{n+1}{n+2} = 1 + \frac{(n+1)(n+1)}{(n+1)(n+2)} = 1 + \frac{n^2 + 2n + 1}{(n+1)(n+2)},$$

so $a_n < a_{n+1}$. The sequence $\{a_n\}$ is also bounded by 2. Actually, $a_n \to 2$.

(2) Let

$$c_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \log n \qquad (n \geq 1).$$

It can be shown that $\{c_n\}$ is a decreasing sequence which is bounded. So it has a limit γ. γ is called Euler's constant. It is important but there is still a lot we do not know about it, e.g., is it rational or irrational? We do know that $\gamma \approx 0.577$.

(3) Let

$$d_n = \left(1 + \frac{1}{n}\right)^n.$$

It can be shown that this sequence is increasing and is bounded by 3. Hence the limit of $\{d_n\}$ exists. It turns out that

$$\lim_{n \to +\infty} d_n = e \approx 2.7183\ldots.$$

Theorem 9.19: Let $\{a_n\}$ be an unbounded sequence. If $\{a_n\}$ is increasing (decreasing) then

$$\{a_n\} \to +\infty \qquad (\{a_n\} \to -\infty).$$

Proof: Not given.

We finally investigate the behaviour of $\{r^n\}$.

Theorem 9.20: Let r be a fixed real number.

(i) If $-1 < r < 1$, then $r^n \to 0$.

(ii) If $r = 1$, then $r^n \to 1$.

(iii) If $r \leq -1$, then $\{r^n\}$ diverges.

(iv) If $r > 1$, then $r^n \to +\infty$.

Proof:

(i) If $-1 < r < 1$, then $|r| < 1$. If $r = 0$, then certainly $r^n \to 0$. So let $r \neq 0$ and let $0 < \epsilon < 1$. Pick a natural number N such that

$$N \geq \frac{\log \epsilon}{\log |r|}.$$

Note that $\log |r| < 0$ and $\log \epsilon < 0$, so

$$N \geq \frac{\log \epsilon}{\log |r|} > 0.$$

Then, if $n > N$, we have

$$\log(|r|^n) = n \log |r| < \frac{\log \epsilon}{\log |r|} \cdot \log |r| = \log \epsilon.$$

Hence $|r|^n < \epsilon$. Therefore we get

$$|r^n - 0| = |r^n| = |r|^n < \epsilon.$$

(ii) This is clear as $r = 1 \Rightarrow r^n = 1$.

(iii) If $r \leq -1$, r^n will oscillate between values ≥ 1 and ≤ -1, so it clearly does not converge.

(iv) If $r > 1$, then $r = 1 + h$ for some $h > 0$. We have

$$r^n = (1+h)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} h^k = 1 + \binom{n}{1} h + \ldots > nh,$$

so if $C > 0$ and $N \geq C/h$, then

$$r^n > nh > \frac{C}{h} h = C \qquad \text{for all} \qquad n > N.$$

Summary: (i) If $a_n = r^n$ then $a_n \to 0$ if $-1 < r < 1$.

(ii) $a_n \to 1$ if $r = 1$.

(iii) Otherwise $\{a_n\}$ diverges.

Examples 9.21:

(1) $3^n \to +\infty$ since $3 > 1$.

(2) $12^{-n} = \left(\frac{1}{12}\right)^n \to 0$ since $-1 < 1/12 < 1$.

(3) $(-1)^n 5^n = (-5)^n$ diverges since $-5 < -1$.

Chapter 10

Infinite Series

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10.1 Introduction

Infinite series are introduced with the specific intention of considering sums of sequences, namely

$$a_0 + a_1 + a_2 + \cdots$$

This is not an entirely straightforward matter, for the sum of infinitely many numbers is as yet completely undefined. What can be defined are the partial sums

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n$$

and the infinite sum must be presumably defined in terms of these partial sums.

10.2 Infinite series

Definition 10.1: The sequence $\{a_n\}$ is summable if the sequence $\{s_n\}$ converges, where

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n.$$

In this case,

$$\lim_{n \to +\infty} s_n$$

is denoted by

$$\sum_{n=0}^{+\infty} a_n,$$

and is called the sum of the sequence $\{a_n\}$.
We also say that

$$\sum_{n=0}^{+\infty} a_n$$

converges (diverges) instead of saying that $\{s_n\}$ converges (diverges).
The most important of all infinite series are the geometric series

$$\sum_{n=0}^{+\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

Only the cases $|r| < 1$ are interesting, since the individual terms do not approach 0 if $|r| \geq 1$, in which case

$$\sum_{n=0}^{+\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

does not converge.
These series can be managed because the partial sums

$$s_n = 1 + r + r^2 + r^3 + \cdots + r^n$$

can be evaluated in simple terms. The two equations

$$s_n = 1 + r + r^2 + r^3 + \cdots + r^n$$

$$rs_n = r + r^2 + r^3 + r^4 + \cdots + r^{n+1}$$

lead to

$$s_n(1 - r) = 1 - r^{n+1}$$

or

$$s_n = \frac{1 - r^{n+1}}{1 - r},$$

where division by $r - 1$ is valid since we assume that $r \neq 1$. It follows that:

Theorem 10.2:

$$\sum_{n=0}^{+\infty} r^n = \lim_{n \to +\infty} s_n = \lim_{n \to +\infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r},$$

since $|r| < 1$. In particular

$$\sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{+\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} = 2.$$

Thus

$$\sum_{n=1}^{+\infty} \left(\frac{1}{2}\right)^n = 1.$$

The sum

$$\sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n$$

with $a_n = (1/2)^n$ has the property that $a_n \to 0$. This turns out to be a necessary condition for summability.

Theorem 10.3: The vanishing condition. If $\{a_n\}$ is summable, if

$$\sum_{n=0}^{+\infty} a_n,$$

exists, then $a_n \to 0$.

Proof: Since $\{a_n\}$ is summable, the partial sums s_n converge to a limit L. Let $\epsilon > 0$. Then there exists an integer N such that

$$n > N \Rightarrow |s_n - L| < \frac{\epsilon}{2}.$$

Hence

$$|a_{n+1}| = |s_{n+1} - s_n| = |(s_{n+1} - L) + (L - s_n)| \le |s_{n+1} - L| + |L - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{for all} \qquad n > N.$$

Unfortunately, the vanishing condition is far from sufficient. For example,

$$\lim_{n \to +\infty} \frac{1}{n} = 0$$

but the sequence $\{1/n\}$ is not summable.

$$\sum_{n=0}^{+\infty} \frac{1}{n}$$

is called the harmonic series.

The harmonic series diverges, though on a computer it looks as though it conveges. For example, to get the sum > 12, one needs 91,390 terms.

Example: Does

$$\sum_{n=0}^{+\infty} \frac{n}{n-1}$$

converge?

No, as $\frac{n}{n-1} \to 1$, the vanishing condition is not satisfied.

If some convergent sums are already available, one can often show convergence of other sums by comparison.

Theorem 10.4: (Comparison Test)

Suppose that $0 \leq a_n \leq b_n$ for all n. Then if

$$\sum_{n=0}^{+\infty} b_n$$

converges, so does

$$\sum_{n=0}^{+\infty} a_n.$$

Proof: Not given.

Theorem 10.5: (Comparison Test, divergence version)

Suppose that $0 \leq a_n \leq b_n$ for all n. Then if

$$\sum_{n=0}^{+\infty} a_n$$

diverges, so does

$$\sum_{n=0}^{+\infty} b_n.$$

Proof: Not given.
Quite frequently the comparison test can be used to analyse very complicated looking series in which most of the complication is irrelevant.

10.2.1 Examples

(1)

$$\sum_{n=0}^{+\infty} \frac{2+\sin^3(n+1)}{2^n+n^2}$$

converges, because

$$0 \leq \frac{2+\sin^3(n+1)}{2^n+n^2} < \frac{3}{2^n}$$

and

$$\sum_{n=0}^{+\infty} \frac{3}{2^n} = 3\sum_{n=0}^{+\infty} \frac{1}{2^n}$$

is a convergent (geometric) series.

(2)

$$\sum_{n=0}^{+\infty} \frac{2^n+5}{3^n+4^n}$$

converges, because

$$\frac{2^n+5}{3^n+4^n} \le \frac{2^n+2^n}{3^n+4^n} \le 2\frac{2^n}{3^n+4^n} \le 2\frac{2^n}{3^n} = 2\left(\frac{2}{3}\right)^n$$

and

$$\sum_{n=0}^{+\infty} 2\left(\frac{2}{3}\right)^n = 2\sum_{n=0}^{+\infty}\left(\frac{2}{3}\right)^n$$

is a convergent (geometric) series.

Theorem 10.6: (Comparison Test, limit form)
If $a_n, b_n > 0$ and

$$\lim_{n\to+\infty}\frac{a_n}{b_n} = c > 0,$$

then

$$\sum_{n=0}^{+\infty} a_n$$

and

$$\sum_{n=0}^{+\infty} b_n$$

both converge, or both diverge to $+\infty$.
Proof: Not given.

Example: Let

$$a_n = \frac{\sqrt{n}}{n^{\frac{3}{2}}+1}.$$

Expect to behave like

$$b_n = \frac{\sqrt{n}}{n^{\frac{3}{2}}} = \frac{1}{n}.$$

As

$$\frac{a_n}{b_n} = \frac{n^{\frac{3}{2}}}{n^{\frac{3}{2}}+1} \to 1 \ne 0$$

as $n \to +\infty$ and

$$\sum_{n=0}^{+\infty} b_n$$

diverges, it follows from Theorem 10.6 that

$$\sum_{n=0}^{+\infty} a_n$$

diverges.
The most important of all tests for summability is the Ratio Test.

Theorem 10.7: Let $a_n > 0$ for all n and suppose that

$$\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = r.$$

Then

$$\sum_{n=0}^{+\infty} a_n$$

converges if $r < 1$.
On the other hand, if $r > 1$, then the terms a_n do not approach 0, so

$$\sum_{n=0}^{+\infty} a_n$$

diverges to $+\infty$.
Notice that it is therefore essential to compute

$$\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} \qquad \text{and NOT} \qquad \lim_{n \to +\infty} \frac{a_n}{a_{n+1}}.$$

Proof: Not given.

10.2.2 Examples

(1) Consider the series

$$\sum_{n=0}^{+\infty} \frac{1}{n!}.$$

Letting $a_n = 1/n!$ we obtain

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

Thus

$$\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = 0.$$

So by the Ratio Test

$$\sum_{n=0}^{+\infty} \frac{1}{n!}$$

converges.

(2) Consider now the series

$$\sum_{n=0}^{+\infty} \frac{r^n}{n!}$$

where $r > 0$ is some fixed real number. Then letting $a_n = r^n/n!$ we obtain

$$\frac{a_{n+1}}{a_n} = \frac{n!}{r^n} \frac{r^{n+1}}{(n+1)!} = \frac{r}{n+1},$$

whence

$$\lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=0.$$

So by the Ratio Test

$$\sum_{n=0}^{+\infty}\frac{r^n}{n!}$$

converges. The vanishing condition yields

$$\lim_{n\to+\infty}\frac{r^n}{n!}=0.$$

(3) Finally, consider the series

$$\sum_{n=0}^{+\infty}nr^n.$$

We have

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)r^{n+1}}{r^n n},$$

so that

$$\lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to+\infty}\frac{n+1}{n}r=r,\qquad\text{since}\qquad\lim_{n\to+\infty}\frac{n+1}{n}=1.$$

This proves that if $0\leq r<1$, then

$$\sum_{n=0}^{+\infty}nr^n$$

converges, and consequently

$$\lim_{n\to+\infty}nr^n=0.$$

Although the Ratio Test will be of the utmost importance as a practical tool it will be frequently be found dissapointing as it appears with maddening regularity that

$$\lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=1.$$

This case is precisely the one which is inconclusive.
For instance, if

$$a_n=\left(\frac{1}{n}\right)^2$$

then

$$\lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to+\infty}\frac{n^2}{(n+1)^2}=1.$$

In fact, our very next test will show that

$$\sum_{n=0}^{+\infty}\left(\frac{1}{n}\right)^2$$

converges.

Theorem 10.8: (The Integral Test)
Suppose that f is positive and decreasing on $[1, +\infty)$ (*i.e.*, $x \leq y \Rightarrow f(x) \geq f(y)$), and that $f(n) = a_n$ for all n. Then

$$\sum_{n=0}^{+\infty} a_n$$

converges if and only if the limit

$$\int_1^{+\infty} dx\ f(x) = \lim_{K \to +\infty} \int_1^K dx\ f(x)$$

exists.
Proof: Not given.

Corollary 10.9: (The p-Test)
If $p > 1$, then

$$\sum_{n=0}^{+\infty} \frac{1}{n^p}$$

converges.
If $0 \leq p \leq 1$, then

$$\sum_{n=0}^{+\infty} \frac{1}{n^p}$$

diverges.
Proof: Not given.

A test closely related to the ratio test is the Root Test. If we try to apply the ratio test to the series

$$\frac{1}{2} + \frac{1}{3} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{3}\right)^3 + \cdots$$

it turns out that the ratios of consecutive terms do not approach a limit. Here the root test works.

Theorem 10.10: (The Root Test)
Supppose that $a_n \geq 0$ and

$$\lim_{n \to +\infty} \sqrt[n]{a_n} = s.$$

Then

$$\sum_{n=0}^{+\infty} a_n$$

converges if $s < 1$ and diverges if $s > 1$.
Proof: Not given.

Proposition 10.11: If

$$\sum_{n=0}^{+\infty} a_n \qquad \text{and} \qquad \sum_{n=0}^{+\infty} b_n$$

converge, then

$$\sum_{n=0}^{+\infty} ca_n \qquad \text{and} \qquad \sum_{n=0}^{+\infty}(a_n + b_n) \qquad \text{and} \qquad \sum_{n=0}^{+\infty}(a_n - b_n)$$

converge also. Moreover

$$\sum_{n=0}^{+\infty} ca_n = c\sum_{n=0}^{+\infty} a_n \qquad \text{and} \qquad \sum_{n=0}^{+\infty}(a_n+b_n) = \sum_{n=0}^{+\infty} a_n+\sum_{n=0}^{+\infty} b_n \qquad \text{and} \qquad \sum_{n=0}^{+\infty}(a_n-b_n)\sum_{n=0}^{+\infty} a_n-\sum_{n=0}^{+\infty} b_n.$$

Proof: Not given.

10.3 Examples

Investigate the convergence of the following series:

(a)

$$\sum_{n=1}^{+\infty} \frac{2n}{5n+3}$$

As

$$\frac{2n}{5n+3} \to \frac{2}{5}$$

the series diverges by the vanishing test.

(b)

$$\sum_{n=0}^{+\infty} 5\left(\frac{1}{4}\right)^n$$

As

$$\sum_{n=0}^{+\infty}\left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

the series converges to $5 \cdot \frac{4}{3} = \frac{20}{3}$.

(c)

$$\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}}$$

Note that

$$\frac{1}{\sqrt{n}} \geq \frac{1}{n},$$

so the series diverges by the Comparison Test.

(d)

$$\sum_{n=1}^{+\infty} \sin\left(\frac{1}{n}\right)$$

Here

$$\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \to \cos 0 = 1.$$

So by the limit version of the comparison test the series diverges.

(e)

$$\sum_{n=1}^{+\infty} \frac{2n-1}{(\sqrt{2})^n}$$

Compute

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2(n+1)-1}{(\sqrt{2})^{n+1}}}{\frac{2n-1}{(\sqrt{2})^n}} = \frac{2n+1}{\sqrt{2}(2n-1)} = \frac{1}{\sqrt{2}}\frac{2n+1}{(2n-1)} \to \frac{1}{\sqrt{2}} \qquad \text{as} \qquad n \to +\infty.$$

So the series converges by the ratio test.

(f)

$$\sum_{n=1}^{+\infty} \frac{1}{(2n+1)^2-1}$$

This series in convergent by the integral test:

$$\int_1^{+\infty} dx \, \frac{1}{(2x+1)^2-1} = \lim_{K\to+\infty}\left(\frac{1}{4}\log\left(\frac{x}{x+1}\right)\Big|_1^K\right) = -\frac{1}{4}\log\left(\frac{1}{2}\right) = \frac{1}{4}\log 2.$$

(g)

$$\sum_{n=1}^{+\infty} \frac{n}{n^4+1}$$

We have

$$\frac{n}{n^4+1} < \frac{1}{n^3}$$

and

$$\sum_{n=1}^{+\infty} \frac{1}{n^3}$$

is convergent as it is a p-series with $p = 3$. We conclude that

$$\sum_{n=1}^{+\infty} \frac{n}{n^4+1}$$

is <u>convergent</u> by the <u>Comparison Test</u>.

(h)

$$\sum_{n=1}^{+\infty} \frac{3^n n!}{n^n}$$

We compute

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \frac{n^n}{3^n n!} = \frac{3(n+1)n^n}{(n+1)^{n+1}} = 3\frac{n^n}{(n+1)^n} = 3\left(\frac{n}{n+1}\right)^n$$

We have

$$\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}.$$

As

$$\left(\frac{1}{1+\frac{1}{n}}\right)^n \to e \qquad \text{as} \quad n \to +\infty$$

we conclude

$$\frac{a_{n+1}}{a_n} = 3\left(\frac{n}{n+1}\right)^n \to \frac{3}{e} > \frac{3}{2.8} > 1.$$

Thus, the series <u>diverges</u> by the <u>ratio test</u>.

(i)

$$\sum_{n=1}^{+\infty} \frac{2^n n!}{n^n}$$

We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \frac{n^n}{2^n n!} = \frac{2(n+1)n^n}{(n+1)^{n+1}} = 2\frac{n^n}{(n+1)^n} = 2\left(\frac{n}{n+1}\right)^n$$

We have

$$\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}.$$

As

$$\left(\frac{1}{1+\frac{1}{n}}\right)^n \to e \qquad \text{as} \quad n \to +\infty$$

we conclude

$$\frac{a_{n+1}}{a_n} = 2\left(\frac{n}{n+1}\right)^n \to \frac{2}{e} < \frac{2}{2.7} < 1.$$

Thus, the series <u>converges</u> by the <u>ratio test</u>.

(j)

$$\sum_{n=1}^{+\infty} \left(\frac{n+1}{4n-1}\right)^n$$

Let

$$a_n = \left(\frac{n+1}{4n-1}\right)^n.$$

Then

$$\sqrt[n]{a_n} = \frac{n+1}{4n-1} = \frac{1+\frac{1}{n}}{4-\frac{1}{n}} \to \frac{1}{4}.$$

Hence the series converges by the root test.

10.4 Series with positive and negative terms

The harmonic series

$$\sum_{n=1}^{+\infty} \frac{1}{n}$$

diverges. It will turn out, however, that the series

$$\sum_{n=1}^{+\infty} (-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \cdots$$

converges. The latter series has positive and negative terms. Other examples for series with positive and negative terms are the power series representations of sin and cos.

Definition 10.12: A series

$$\sum_{n=0}^{+\infty} a_n$$

converges absolutely (is absolutely convergent) if

$$\sum_{n=0}^{+\infty} |a_n|$$

converges.

10.4.1 Examples

(1) Every convergent series

$$\sum_{n=0}^{+\infty} a_n$$

with $a_n \geq 0$ is absolutely convergent.

(2)

$$\sum_{n=0}^{+\infty} (-1)^n \frac{1}{2^n}$$

is absolutely convergent because

$$\sum_{n=0}^{+\infty} \frac{1}{2^n}$$

is convergent.
Note that we have not assumed

$$\sum_{n=0}^{+\infty} a_n$$

to be convergent. This is not necessary, as the following result shows.

Proposition 10.13: If

$$\sum_{n=1}^{+\infty} a_n$$

is absolutely convergent, then

$$\sum_{n=1}^{+\infty} a_n$$

is convergent.
Proof: Not given.

10.4.2 Examples

(a) The series

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n!}$$

converges (absolutely).
Proof: We have

$$|a_n| = \frac{1}{n!} \Rightarrow \frac{|a_{n+1}|}{|a_n|} = \frac{1}{n+1} \to 0.$$

Thus the series

$$\sum_{n=1}^{+\infty} |a_n|$$

converges by the Ratio Test and hence

$$\sum_{n=1}^{+\infty} a_n$$

converges by Proposition 10.13.
(b) The series

$$\sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{n^2}$$

converges (absolutely).

Proof: The ratio test does not work here as

$$|a_n| = \frac{1}{n^2} \Rightarrow \frac{|a_{n+1}|}{|a_n|} = \frac{n^2}{(n+1)^2} \to 1.$$

But we have already shown that

$$\sum_{n=1}^{+\infty} |a_n| = \sum_{n=1}^{+\infty} \frac{1}{n^2}$$

converges by the p-test. Hence

$$\sum_{n=1}^{+\infty} a_n$$

converges by Proposition 10.13.
There are series which are convergent but not absolutely convergent (such series are sometimes called conditionally convergent.

Leibniz's Theorem 10.14: Supppose that

$$a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$$

and that

$$\lim_{n\to+\infty} a_n = 0.$$

Then, the series

$$\sum_{n=1}^{+\infty} (-1)^{n+1} a_n$$

converges.
Proof: Not given.

10.4.3 Examples

(i)

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}$$

Solution: Let $a_n = 1/n$. Then

$$a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$$

and

$$\lim_{n\to+\infty} a_n = 0.$$

So the series converges by Leibniz's theorem.

(ii)

$$\sum_{n=1}^{+\infty}(-1)^{n+1}\left(\frac{n}{2n+1}\right)$$

Solution: Let $a_n = n/(2n+1)$. Then

$$\lim_{n\to+\infty} a_n = \frac{1}{2},$$

this series does not converge due to the vanishing test.

(iii)

$$\sum_{n=1}^{+\infty}(-1)^{n+1}\left(\frac{1}{5}\right)^n$$

Solution: Let $a_n = (1/5)^n$. Clearly

$$a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$$

and

$$\lim_{n\to+\infty} a_n = 0.$$

So the series converges by Leibniz's theorem. However, this series converges absolutely too, using the ratio test.

Chapter 11

Power series

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11.1 Introduction

Definition 11.1: A power series about the point a is an expression of the form

$$a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$$

in which the coefficients a_n of the power series are real numbers and x is a variable.

For any fixed numerical value of x this infinite series in powers of $x-a$ will become an infinite series of the type considered in the previous chapter, and so will either converge or diverge. Thus a power series in $x-a$ will define a function of x for all x in the interval in which the series converges.

In terms of the summation notation we can write the power series as

$$\sum_{n=0}^{+\infty} a_n(x-a)^n,$$

and if the function to which this infinite series converges is denoted by $f(x)$, often called its sum function, we may write

$$f(x) = \sum_{n=0}^{+\infty} a_n(x-a)^n.$$

The interval in which this power series will converge will depend on the coefficients a_n, the point a about which the series is expanded and x itself.

Theorem 11.2: For any power series

$$\sum_{n=0}^{+\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$$

one of the following three possibilities must be true:

1. The sum
$$\sum_{n=0}^{+\infty} a_n(x-a)^n$$
converges only for $x = a$.

2. The sum
$$\sum_{n=0}^{+\infty} a_n(x-a)^n$$
converges absolutely for all x.

3. There is a number $R > 0$ such that
$$\sum_{n=0}^{+\infty} a_n(x-a)^n$$
converges absolutely when $|x-a| < R$ and diverges when $|x-a| > R$.

   Notice that we do not mention what happens when $|x-a| = R$.

Proof: Not given. □

Definition 11.3: The number R which occurs in case (3) of 11.2 is called the radius of converge of the sum

$$\sum_{n=0}^{+\infty} a_n(x-a)^n.$$

In case (1) and (2) it is customary to say that the radius of convergence is 0 and $+\infty$, respectively.

When $0 < R < +\infty$, the interval $(a-R, a+R)$ is called the interval of convergence.

11.1.1 Examples

(1) $\sum_{n=0}^{+\infty} n!(x-a)^n$ $(a_n = n!)$ converges only for $x = a$, for if $x \neq a$, then

$$\left|\frac{(n+1)!(x-a)^{n+1}}{n!(x-a)^n}\right| = (n+1)|x-a| \to +\infty \qquad \text{as} \qquad n \to +\infty,$$

entailing divergence by the proof of the Ratio Test.

(2) $\sum_{n=0}^{+\infty} \frac{(x-a)^n}{n!}$ ($a_n = 1/n!$) converges absolutely for all $x \in \mathbb{R}$ by the Ratio Test, because

$$\frac{\left|\frac{(x-a)^{n+1}}{(n+1)!}\right|}{\left|\frac{(x-a)^n}{n!}\right|} = \frac{|x-a|}{n+1} \to 0.$$

(3) The radius of convergence of

$$\sum_{n=0}^{+\infty} (x-a)^n$$

(here $a_n = 1$) is 1 by the Ratio Test since

$$\lim_{n\to+\infty} \left|\frac{(x-a)^{n+1}}{(x-a)^n}\right| = |x-a|.$$

11.2 Taylor and Maclaurin series

Many elementary functions can be defined via power series, *e.g.,*

$$\begin{aligned}
e^x &= \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\sin x &= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\cos x &= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\end{aligned}$$

and many nice functions have a power series expansion. To see this we need to know how power series can be differentiated.

Theorem 11.4: If

$$\sum_{n=0}^{+\infty} a_n x^n$$

has radius R and sum $f(x)$ ($|x| < R$), then the series

$$\sum_{n=0}^{+\infty} (n+1)a_{n+1}x^n = \sum_{n=1}^{+\infty} n a_n x^{n-1}$$

obtained by differentiating

$$\sum_{n=0}^{+\infty} a_n x^n$$

term by term, also has radius R and sum $f'(x)$ ($|x| < R$).
Proof: Not given.

11.2.1 Example

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \qquad (|x| < 1).$$

So

$$1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots = \frac{1}{(1-x)^2} \qquad (|x| < 1).$$

and

$$2 + 3 \cdot 2x + 4 \cdot 3x^2 + \cdots + n(n-1)x^{n-2} + \cdots = \frac{2}{(1-x)^3} \qquad (|x| < 1).$$

Here is a strategy for representing a given function in the form of a power series expanded about a point a.

Suppose f is a function which may be differentiated arbitrarily many times at a. To represent $f(x)$ in the form of the power series

$$f(x) = \sum_{n=0}^{+\infty} a_n (x-a)^n,$$

we need to determine the coefficients a_n. Differentiating

$$f(x) = \sum_{n=0}^{+\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$$

term by term, we obtain

$$f'(x) = f^{(1)}(a) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 \cdots + na_n(x-a)^{n-1} + \cdots$$

and we can conclude that

$$f^{(1)}(a) = a_1.$$

Differentiating the series

$$a_1 + 2a_2(x-a) + 3a_3(x-a)^2 \cdots + na_n(x-a)^{n-1} + \cdots$$

we obtain

$$f^{(2)}(x) = 2a_2 + 3 \cdot 2a_3(x-a) + 4 \cdot 3a_4(x-a)^2 \cdots + n(n-1)a_n(x-a)^{n-2} + \cdots$$

hence

$$a_2 = \frac{f^{(2)}(a)}{2!}.$$

Continuing in this way we get

$$a_n = \frac{f^{(n)}(a)}{n!} \qquad \text{for all} \qquad n \geq 0.$$

Note that we also have

$$f^{(0)}(a) = f(a) = a_0$$

and by convention $0! = 1$.

Substituting these coefficients into the original series gives

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots = \sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Definition 11.5: This power series expansion of f is called the Taylor series expansion of f about the point a.

When $a = 0$, the Taylor series expansion reduces to the Maclaurin series expansion of f (a Maclaurin series is always an expansion about the origin).

11.2.2 Example

(1) Find the Maclaurin series expansion of $\sin x$.
Solution: We have

$$\begin{aligned}
\sin^{(0)}(0) &= \sin 0 = 0\ ,\\
\sin^{(1)}(0) &= \cos 0 = 1\ ,\\
\sin^{(2)}(0) &= -\sin 0 = 0\ ,\\
\sin^{(3)}(0) &= -\cos 0 = -1\ .
\end{aligned}$$

Thus, we see that $\sin^{(n)}(0) = 0$ if n is even, but that $f^{(n)}(0)$ alternates between 1 and -1 when n is odd. Substituting the values of $f^{(n)}(0)$ into the general Maclaurin series ($a = 0$) shows that the Maclaurin series for $\sin x$ is

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

(2) Find the Maclaurin series expansion of e^x.
Solution: We set $f(x) = e^x$ and use the fact that $f'(x) = f(x)$, we find that

$$\begin{aligned}
f^{(0)}(0) &= 1\ ,\\
f^{(1)}(0) &= 1\ ,\\
f^{(2)}(0) &= 1\ ,\\
f^{(3)}(0) &= 1\ .
\end{aligned}$$

Thus, we see that $f^{(n)}(0) = 1$. Substituting the values of $f^{(n)}(0)$ into the general Maclaurin series ($a = 0$) shows that the Maclaurin series for f is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

Replacing x by $-x$ we see that the Maclaurin series for e^{-x} is

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{+\infty} (-1)^n \frac{x^n}{n!}.$$

(3) Find the Maclaurin series expansion of $g(x) = \log(1+x)$.
Solution: We have

$$\begin{aligned} g^{(0)}(0) &= 0\ , \\ g^{(1)}(x) &= \frac{1}{1+x},\ g^{(1)}(0) = 1\ , \\ g^{(2)}(x) &= \frac{-1}{(1+x)^2},\ g^{(2)}(0) = -1\ , \\ g^{(3)}(x) &= \frac{1\cdot 2}{(1+x)^3},\ g^{(3)}(0) = 2 = 2!\ . \end{aligned}$$

In general

$$g^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n},\ g^{(n)}(0) = (-1)^{n+1}(n-1)!$$

for $n = 1, 2, 3, \cdots$.
Thus, the Maclaurin series for $g(x)$ is

$$g(x) = \log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

11.2.3 Examples

(1) Find the Taylor series expansion of $\cos x$ about the point a.
Solution: Set $f(x) = \cos x$. We have to compute the derivatives:

$$\begin{aligned} f^{(0)}(a) &= \cos a\ , \\ f^{(1)}(x) &= -\sin x,\ f^{(1)}(a) = -\sin a\ , \\ f^{(2)}(x) &= -\cos x,\ f^{(2)}(a) = -\cos a\ , \\ f^{(3)}(x) &= \sin x,\ f^{(3)}(a) = \sin a\ . \end{aligned}$$

Hereafter, further differentiation simply repeats this pattern of coefficients. Thus the Taylor series expansion of $\cos x$ about the point a is

$$\cos x = \cos a - \frac{\sin a}{1!}(x-a) - \frac{\cos a}{2!}(x-a)^2 + \frac{\sin a}{3!}(x-a)^3 + \frac{\cos a}{4!}(x-a)^4 - \cdots$$

$$= \cos a \sum_{n=0}^{+\infty} (-1)^n (x-a)^{2n} \frac{1}{(2n)!} - \sin a \sum_{n=0}^{+\infty} (-1)^n (x-a)^{2n+1} \frac{1}{(2n+1)!}$$

as the series can be shown to be absolutely convergent. Absolutely convergence entails that we may rearrange terms without altering its sum.

As a special case, by setting $a = 0$, we obtain from this the Maclaurin series expansion of $\cos x$

$$\cos x = \sum_{n=0}^{+\infty} (-1)^n x^{2n} \frac{1}{(2n)!}.$$

(2) Find the Taylor series expansion of $f(x) = (2+x)^{\frac{-1}{2}}$ about the point $a = 1$.
Solution:
Set $f(x) = (2+x)^{-\frac{1}{2}} = \left(\frac{1}{2+x}\right)^{\frac{1}{2}}$. We must set $a = 1$ and then compute the derivatives:

$$\begin{aligned}
f^{(0)}(x) &= f(x) = \left(\frac{1}{2+x}\right)^{\frac{1}{2}}, \quad \text{so} \quad f^{(0)}(1) = \left(\frac{1}{3}\right)^{\frac{1}{2}}, \\
f^{(1)}(x) &= -\frac{1}{2} \cdot \left(\frac{1}{2+x}\right)^{\frac{3}{2}}, \quad \text{so} \quad f^{(1)}(1) = -\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{3}{2}}, \\
f^{(2)}(x) &= \frac{1}{2} \cdot \frac{3}{2}\left(\frac{1}{2+x}\right)^{\frac{5}{2}}, \quad \text{so} \quad f^{(2)}(1) = \frac{1 \cdot 3}{2^2}\left(\frac{1}{3}\right)^{\frac{5}{2}}, \\
f^{(3)}(x) &= -\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}\left(\frac{1}{2+x}\right)^{\frac{7}{2}}, \quad \text{so} \quad f^{(3)}(1) = -\frac{1 \cdot 3 \cdot 5}{2^3}\left(\frac{1}{3}\right)^{\frac{7}{2}}.
\end{aligned}$$

Hereafter, inspection of the general pattern of the results shows that

$$f^{(n)}(1) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n}\left(\frac{1}{3}\right)^{\frac{(2n+1)}{2}}.$$

Substituting the values for $f^{(n)}(1)$ into the general form of the Taylor series and setting $a = 1$, shows that the required Taylor series expansion of $(2+x)^{-\frac{1}{2}}$ about the point 1 is

$$\left(\frac{1}{2+x}\right)^{\frac{1}{2}} = \left(\frac{1}{3}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{3}{2}} \frac{(x-1)}{1!} + \frac{1 \cdot 3}{2^2}\left(\frac{1}{3}\right)^{\frac{5}{2}} \frac{(x-1)^2}{2!}$$
$$-\frac{1 \cdot 3 \cdot 5}{2^3}\left(\frac{1}{3}\right)^{\frac{7}{2}} \frac{(x-1)^3}{3!} + \dots + (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n}\left(\frac{1}{3}\right)^{\frac{(2n+1)}{2}} \cdot \frac{(x-1)^n}{n!} + \cdots.$$

So far, our development of functions in terms of Taylor and Maclaurin series has been formal, in the sense that although we now know how to relate a power series to a given function, we have not actually proved that the function and its series are equal.

The connection is known as Taylor's theorem.

11.3 Taylor's theorem

Taylor's theorem If f is differentiable n times on an open interval I and $f^{(n)}$ is continuous on I and $a \in I$, then $f(x)$ can be written

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + R_n(x)$$

for all $x \in I$, where the remainder term is

$$R_n(x) = \frac{f^{(n)}(\xi)}{n!}(x-a)^n,$$

with ξ some number between a and x.

The above result reduces to the corresponding Maclaurin series expansion and remainder term when $x = 0$.
Proof: Not given.

The polynomial

$$T_n(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the Taylor polynomial of degree n obtained when f is expanded about the point a.

Corollary: If f is differentiable infinitely many times on an open interval I and $a, x \in I$ and

$$\lim_{n \to +\infty} R_n(x) = 0,$$

then

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$= \sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Proof: Not given.

11.3.1 Example

Find the Taylor polynomial of degree 3 which approximates the function $e^{-x/2}$ when the expansion is about the point $a = 1$. Determine the magnitude of the error involved when x is in the interval $-2 \leq x \leq 2$, *i.e.,* $R_4(x)$.
Solution: We set $f(x) = e^{-x/2}$ and $a = 1$. Then, we have

$$\begin{aligned}
f^{(0)}(x) &= e^{-x/2}, \; f^{(0)}(1) = e^{\frac{-1}{2}} , \\
f^{(1)}(x) &= -\frac{1}{2}e^{-x/2}, \; f^{(1)}(1) = -\frac{1}{2}e^{\frac{-1}{2}} , \\
f^{(2)}(x) &= \frac{1}{2^2}e^{-x/2}, \; f^{(2)}(1) = \frac{1}{4}e^{\frac{-1}{2}} , \\
f^{(3)}(x) &= -\frac{1}{2^3}e^{-x/2}, \; f^{(3)}(1) = -\frac{1}{8}e^{\frac{-1}{2}} , \\
f^{(4)}(x) &= \frac{1}{2^4}e^{-x/2} .
\end{aligned}$$

Thus the Taylor polynomial of degree 3 about the point $a = 1$ is

$$T_3(x) = e^{\frac{-1}{2}} - \frac{1}{2}e^{\frac{-1}{2}}(x-1) + \frac{1}{8}e^{\frac{-1}{2}}(x-1)^2 - e^{\frac{-1}{2}}\frac{1}{48}(x-1)^3.$$

and the remainder is

$$R_4(x) = \frac{f^{(4)}(\xi)}{4!}(x-1)^4 = \frac{1}{16}e^{-\xi/2}\frac{(x-1)^4}{4!}$$

with ξ some number between 1 and x. in the interval $\xi \in [-2, 2]$ and $\xi \neq 1$.

To estimate the magnitude of the error made when $e^{-x/2}$ is represented by $T_3(x)$ with $-2 \leq x \leq 2$, we proceed as follows. The function $e^{-x/2}$ is a strictly decreasing function of x, so on the interval $-2 \leq x \leq 2$ its maximum value occurs at the left end point of the interval where it equals e. The maximum value of the non-negative function $(x-1)^4$ on the interval $-2 \leq x \leq 2$, occurs at the left end point at which it equals $(-3)^4 = 81$. Thus, we have the estimate

$$R_4(x) \leq \frac{e}{16 \cdot 4!} \cdot 81 = 0.57339.$$